MATH 521A: Abstract Algebra

Homework 2: Due Sep. 9 (Wednesday)

1. Find all primes between 1000 and 1050.
2. Let $a = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$ and $b = p_1^{s_1} p_2^{s_2} \cdots p_k^{s_k}$ where $p_1, \ldots, p_k$ are distinct positive prime integers, and each $r_i, s_i \in \mathbb{N}_0$. Prove that $a|b$ if and only if $\forall i \in [1, k], r_i \leq s_i$.
3. Let $a = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$ and $b = p_1^{s_1} p_2^{s_2} \cdots p_k^{s_k}$ where $p_1, \ldots, p_k$ are distinct positive prime integers, and each $r_i, s_i \in \mathbb{N}_0$. Determine, with proof, the prime factorization of $\gcd(a, b)$ and $\text{lcm}(a, b)$.
4. Let $a, b, m, n \in \mathbb{N}$. Prove that $a^m | b^m$ if and only if $a^n | b^n$.
5. Prove that, for all $n \geq 2$, there are no primes among $\{n! + 2, n! + 3, \ldots, n! + n\}$.
6. Prove that, for integer a, b and prime p:

   $$ab \equiv 0 \pmod p \text{ if and only if } [\ a \equiv 0 \pmod p \text{ or } b \equiv 0 \pmod p\ ]$$

   Now assume p is composite and disprove the statement.
7. Prove that, for integer a, b and prime p:

   $$a^2 \equiv b^2 \pmod p \text{ if and only if } [\ a \equiv b \pmod p \text{ or } a \equiv -b \pmod p\ ]$$

   Now find a composite p and a, b to disprove the statement.
8. Let $a, b, c, n \in \mathbb{N}$. Prove that $a \equiv b \pmod n$ if and only if $ac \equiv bc \pmod{nc}$.
9. Let $a, b, n \in \mathbb{N}$. Determine the exact conditions under which the modular equation

   $$ax \equiv b \pmod{an}$$

   has solution(s) (for x).
10. Let $a, b, m, n \in \mathbb{N}$. Prove that:

    $$[\ a \equiv b \pmod m \text{ and } a \equiv b \pmod n\ ] \text{ if and only if } a \equiv b \pmod{\text{lcm}(m, n)}$$