MATH 521A: Abstract Algebra
Homework 1: Due Aug. 31

1. Prove that $(-\mathbb{N}_0)$, the set of nonpositive integers, is well-ordered.

   For a set T, we say it is *inductively ordered* if there is some special $t \in T$ and some function $f : T \to T$ such that:
   (1) The elements $t, f(t), f(f(t)), \ldots$ are all distinct; and
   (2) $T = \{t, f(t), f(f(t)), \ldots\}$.

2. Prove that $\mathbb{N}_0$ is inductively ordered.

3. Prove that if a set is inductively ordered then it is well-ordered.

4. Prove that the square of any integer a is either of the form $4k$ or of the form $4k + 1$ for some integer k.

5. Prove the *Backwards Division Algorithm*: Let a, b be integers with $b > 0$. Then there exist integers q, r such that $a = bq + r$ with $-b < r \leq 0$.

6. Let $a, b \in \mathbb{N}$ with $a|b$. Prove that $a \leq b$.

7. Let a, b be nonzero integers with $a|b$ and $b|a$. Prove that $a = \pm b$.

8. Let $a, b \in \mathbb{Z}$, not both zero, and let $d = \gcd(a, b)$. Prove that d divides each element of $S = \{am + bn : m, n \in \mathbb{Z}\}$.

9. Use the Euclidean Algorithm to find $\gcd(175, 630)$ and to express this gcd as a linear combination of $175, 630$.

10. Prove that $\gcd(a, b) = \gcd(a, b + at)$, for every $t \in \mathbb{Z}$.

Warning: For these problems, do not use the Fundamental Theorem of Arithmetic (unique factorization of integers into primes).