MATH 521A: Abstract Algebra Exam 2

Please read the following instructions. For the following exam you are free to use a calculator and any papers you like, but no books or computers. Please turn in **exactly four** problems. You must do problems 1-3, and one more chosen from 4-6. Please write your answers on separate paper, make clear what work goes with which problem, and put your name or initials on every page. You have 50 minutes. Each problem will be graded on a 5-10 scale (as your quizzes), for a total score between 20 and 40. This will then be multiplied by $\frac{5}{2}$ for your exam score.

Turn in problems 1,2,3:

1. Let $R = \mathbb{Z}, U = 5\mathbb{Z}$, two rings. Suppose $U \subseteq V \subseteq R$, and V is a ring. Prove that $V = U$ or $V = R$.

2. For ring R, and $x \in R$, define the *centralizer of* x, as
$$C_x(R) = \{a \in R : ax = xa\}.$$
Prove that $C_x(R)$ is a subring of R.

3. Let S be the ring of all continuous real-valued functions defined on $[0, 1]$, with the natural ring operations $(f \oplus g)(x) = f(x) + g(x), \quad (f \odot g)(x) = f(x)g(x)$. Define $\phi : S \to \mathbb{R}$ as $\phi : f \mapsto f(1/2)$. Prove that ϕ is a homomorphism, and find its kernel and image.

Turn in exactly one more problem of your choice:

4. Prove that $\mathbb{Q}[\sqrt[3]{2}] = \{a + b\sqrt[3]{2} + c\sqrt[3]{4} : a, b, c \in \mathbb{Q}\}$ is a commutative ring with identity.

5. Let $X = \{1, 2, 3, 4, 5\}$, and let the power set of X, denoted $\mathcal{P}(X)$, be the set of all subsets of X. Let R have ground set $\mathcal{P}(X)$, with operations $a \odot b = a \cap b$ and
$$a \oplus b = a\Delta b = (a \setminus b) \cup (b \setminus a) = (a \cup b) \setminus (a \cap b)$$
Prove that R is a commutative ring with identity.

6. For ring R, $x \in R$, and $n \in \mathbb{N}$, we say x has *additive order* n if $\underbrace{x + x + \cdots + x}_{n} = 0_R$, and for $m < n$ we have $\underbrace{x + x + \cdots + x}_{m} \neq 0_R$. Define $T \subseteq R$ to be the set of those elements of R that have an additive order. Prove that T is a subring of R.

You may also turn in the following (optional):

7. Describe your preferences for your next group assignment. (will be kept confidential)