Math 254 Spring 2014 Exam 8 Solutions

1. Carefully state the definition of the "standard vector space" $\mathbb{R}^n$. Give a set of two vectors from $\mathbb{R}^4$.

   $\mathbb{R}^n$ is the set of all ordered n-tuples (or lists) of real numbers. A possible set requested is $\{(1,2,3,4),(4,3,2,1)\}$.

2. Let F, G be the linear mappings on $\mathbb{R}^2$ defined by $F(x,y) = (y, x+y)$ and $G(x,y) = (0, 2x)$. Find formulas defining the mappings: (a) $F+G$; (b) $2F-G$; (c) FG; (d) GF; (e) F^2

   Note that in all cases solutions will be functions from $\mathbb{R}^2$ to $\mathbb{R}^2$.
   (a) $(F+G)(x,y) = (y, 3x+y)$; (b) $(2F-3G)(x,y) = (2y, 2y)$; (c) $(FG)(x,y) = F(G(x,y)) = F(0,2x) = (2x,2x)$; (d) $(GF)(x,y) = G(F(x,y)) = G(y,x+y) = (0,2y)$; (e) $F^2(x,y) = F(F(x,y)) = F(y,x+y) = (x+y, x+2y)$

The remaining three problems are all in vector space $V = \mathbb{R}^2$, with function $F: V \to V$ given by $F(x,y) = (x+2y, 3x+4y)$.

3. Prove that F satisfies the definition of a linear mapping.

   We need to prove two properties for F:
   1) $F((x,y)+(x',y')) = F(x+x', y+y') = (x+x'+2(y+y'), 3(x+x')+4(y+y')) = (x+2y, 3x+4y) + (x'+2y', 3x'+4y') = F(x,y) + F(x',y')$
   2) $F(k(x,y)) = F(kx,ky) = (kx+2ky, 3kx+4ky) = k(x+2y, 2x+4y) = kF(x,y)$

4. Find the rank and nullity of F.

   You need any two of the following three ideas:

   Nullity: If $F(x,y) = (0,0)$ then $(x+2y, 3x+4y) = (0,0)$ which gives two equations $x+2y=0, 3x+4y=0$. These have unique solution $x=y=0$, so the kernel of F is 0-dimensional and the nullity is 0.

   Rank: $F(1,0) = (1,3)$ and $F(0,1) = (2,4)$. The rank of F is the dimension of $Span(S)$, for $S = \{(1,3),(2,4)\}$. Since S is independent, the rank of F is 2.

   Rank-Nullity Theorem: The domain of F is $\mathbb{R}^2$, which has dimension 2, so the rank of F plus the nullity of F equals 2.

5. Determine whether F is an isomorphism, and if so find its inverse.

   Because the nullity of F is 0, F is an isomorphism. Here are two ways to finish:

   Method 1: We want $(r,s) = (x+2y, 3x+4y)$, or $x+2y=r, 3x+4y=s$. We solve these to get $x = s-2r, y = \frac{3r-s}{2}$, so $F^{-1}(r,s) = (s-2r, \frac{3r-s}{2})$.

   Method 2: $F(x,y) = A(x,y)^T$, for $A = \left(\begin{smallmatrix}1 & 2\\ 3 & 4\end{smallmatrix}\right)$. Hence $F^{-1}(x,y) = A^{-1}(x,y)^T$. We calculate $A^{-1} = \frac{1}{-2}\left(\begin{smallmatrix}4 & -2\\ -3 & 1\end{smallmatrix}\right) = \left(\begin{smallmatrix}-2 & 1\\ 1.5 & -0.5\end{smallmatrix}\right)$, so $F^{-1}(x,y) = (-2x+y, 1.5x-0.5y)$.