Math 254 Spring 2014 Exam 2a Solutions

1. Carefully state the definition of the "polynomial space" $P(t)$. Give two example vectors from $P_1(t)$.

   $P(t)$ is the vector space consisting of all polynomials in the variable t. Two vectors from $P_1(t)$ are $3, 3+2t$.

2. List, in any order, the three elementary operations that leave unchanged the solution set to a system of linear equations.

   E1: Interchange two equations. E2: Multiply an equation by a *nonzero* constant. E3: Add a multiple of one equation to another.

3. Solve the following system of equations using Gaussian elimination and back-substitution.

   $$\begin{array}{rcl} 2y-z &=& 1 \\ x-y+z &=& 1 \\ 2x+y+2z &=& 2 \end{array} \qquad \begin{array}{rcl} x-y+z &=& 1 \\ 2y-z &=& 1 \\ 2x+y+2z &=& 2 \end{array} \qquad \begin{array}{rcl} x-y+z &=& 1 \\ 2y-z &=& 1 \\ 3y+0z &=& 0 \end{array} \qquad \begin{array}{rcl} x-y+z &=& 1 \\ 2y-z &=& 1 \\ 1.5z &=& -1.5 \end{array}$$

   Step 1: $E_1 \leftrightarrow E_2$. Step 2: $-2E_1+E_3 \to E_3$. Step 3: $-1.5E_2+E_3 \to E_3$. We now back-substitute: $z=-1$, then $2y+1=1$ so $y=0$. Lastly $x-0-1=1$ so $x=2$.

4. Consider the system of equations $\{2x-2y=4, 4x+ay=b\}$. For which values of a, b does this have exactly one solution (and what is it)? For which values of a, b does this have no solution? For which values of a, b does this have infinitely many solutions?

   We solve: $-2E_1+E_2 \to E_2$ gives us $\{2x-2y=4, (4+a)y=(-8+b)\}$, back-substitute $y=\frac{b-8}{a+4}$ and $2x-2\frac{b-8}{a+4}=4$ so $x=2+\frac{b-8}{a+4}=\frac{2a+b}{a+4}$. This is a unique solution, provided $a \neq -4$. If $a=-4$, then the second equation is $0=b-8$. If $a=-4$ and $b=8$, there are infinitely many solutions; if $a=-4$ and $b \neq 8$, then there are no solutions.

5. Find a set of points in the plane that have infinitely many lines of best fit. Be sure to justify your answer.

   One solution is any single point, e.g. $\{(2,3)\}$. This gives system $\{b+2m=3, 2b+4m=6\}$, which has infinitely many solutions since the second equation is twice the first. Another solution is the empty set, which gives system $\{0b+0m=0, 0b+0m=0\}$. Others are possible, e.g. $\{(0,3),(0,5)\}$, with system $\{2b+0m=8, 0b+0m=0\}$.