Math 254 Spring 2014 Exam 1 Solutions

1. Carefully state the definition of "linear function space". Give two example vector spaces.

   A linear function space is the (linear) span of some set of variables. Examples include $\mathrm{Span}(x)$, $\mathrm{Span}(x, y, z)$, $\mathrm{Span}(s, t)$.

2. State whether or not each of the following are defined; if defined state what type of object is the result. Note: "3-vector" means "vector from $\mathbb{R}^3$".

   (a) (column 3-vector)(row 3-vector)

   Juxtaposition means matrix multiplication. The result is a 3×3 matrix.

   (b) (column 3-vector)$\cdot$(row 3-vector)

   $\cdot$ means dot product (of vectors). The result is a scalar.

   (c) (column 3-vector)$\times$(row 3-vector)

   $\times$ means cross product (of 3-vectors only). The result is a 3-vector.

   (d) (3-vector)+(3-vector)

   Two vectors from the same vector space may always be added. The result is a 3-vector.

   (e) (3×1 matrix)+(1×3 matrix)

   Matrices of different dimensions may not be added; this sum is not defined.

3. Let $u = (1, 1, 1, 3), v = (-1, 0, x, 1)$. Determine all possible x, if any, that make vectors $u, v \in \mathbb{R}^4$ orthogonal.

   We calculate $u \cdot v = -1 + 0 + x + 3 = 2 + x$. The vectors u, v are orthogonal exactly when $u \cdot v = 0$, that is when $2 + x = 0$, or $x = -2$.

4. Let $u = (1, 2, 3)$. Find any vector v such that $u \times v = (0, 0, 0)$.

   We have $\|u \times v\| = \|u\|\|v\| \sin\theta$, which we want to be $\|(0, 0, 0)\| = 0$. Hence we want $\theta = 0$, or v to be parallel to u. Thus any multiple of u will work: v can be $(1, 2, 3)$ or $(2, 4, 6)$ or $(-1, -2, -3)$ or $(0, 0, 0)$.

5. Calculate the matrix product $\begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix} \begin{bmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{bmatrix}$. Hint: it's nice.

   We calculate $\begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix} \begin{bmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{bmatrix} = \begin{bmatrix} 1/4+3/4 & -\sqrt{3}/4+\sqrt{3}/4 \\ \sqrt{3}/4-\sqrt{3}/4 & 3/4+1/4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Later we will learn that this isn't a surprise; we have demonstrated that rotating counterclockwise by an angle of $-\frac{2\pi}{3}$, followed by rotating counterclockwise by an angle of $\frac{2\pi}{3}$, is the same as not doing anything at all.