Behold the most important ideas of the course, in bold. Please memorize them; they will be tested on every exam. Further, you need to understand them in all their intricacies – you should be able to provide examples and determine whether an object you are given meets a particular definition or not. A definition is a sentence and must satisfy all ordinary rules of English grammar. Generally each noun, verb, and adjective in a definition is essential and omitting even one of these would not be correct.

1. A vector space is a collection of objects called vectors, together with a way to add vectors and multiply by real numbers (called scalars). These latter properties are together called closure; two equivalent statements are given in the comments. We normally denote the vector space with upper case letters like \( V, U, W \), and the vectors themselves with lower case letters like \( u, v, v_1, v' \). Sometimes to emphasize that they are vectors we will put a bar or arrow over the top as \( \vec{v} \) or \( \vec{u} \).

2. For any set of vectors \( \{v_1, v_2, \ldots, v_k\} \), their span is the set \( \{a_1v_1 + a_2v_2 + \cdots + a_kv_k\} \), where each of \( a_1, a_2, \ldots, a_k \) varies over every real number. Note that the span is also a set of vectors, which is a subset of the vector space from which the original set is drawn. We denote it as \( \text{Span}(v_1, v_2, \ldots, v_k) \), and call the elements of this set linear combinations of \( v_1, v_2, \ldots, v_k \). More compactly, we write \( \text{Span}(v_1, v_2, \ldots, v_k) = \{a_1v_1 + a_2v_2 + \cdots + a_kv_k : a_1, a_2, \ldots, a_k \in \mathbb{R}\} = \{\sum_{i=1}^k a_i v_i : a_i \in \mathbb{R}\} \).

The next five definitions are the most important examples of vector spaces, at least in this course.

3. The linear function space in a set of variables \( \{x_1, x_2, \ldots, x_k\} \) is their span, or (using the above notation) \( \text{Span}(x_1, x_2, \ldots, x_k) \). Note: the vectors in this vector space are linear functions, such as \( 3x \) or \( 4x - 2y \). Note that a linear function may NOT include a constant, e.g. \( 4x + 5y + 3 \) is not linear.

4. The polynomial space in a variable \( t \), denoted \( P(t) \), is the set of all polynomials in the single variable \( t \). Note: the vectors in this vector space are polynomials, like \( 2 + t \) or \( 3 + 7t - 4t^5 \). Often we prefer a subset of this space, by restricting to a maximum degree \( n \), which we denote \( P_n(t) \). For example, \( 6t^2 + 3t - 4 \) and \( -4t^2 + 8 \) are both in \( P(t) \), and also \( P_2(t), P_3(t), \ldots \). Neither is in \( P_1(t) \) or \( P_0(t) \).

5. For any positive integers \( m, n \), the matrix space \( M_{m,n} \) is the set of all matrices with \( m \) rows and \( n \) columns (with real numbers as entries). Note: the vectors in this vector space are matrices, like \( \begin{pmatrix} 1 & 2 \end{pmatrix} \). If \( m = n \) we say the matrix is square, and sometimes abbreviate \( M_{n,n} \) as \( M_n \).

6. For any positive integer \( n \), the standard vector space \( \mathbb{R}^n \) is the set of all \( n \)-tuples of real numbers. Note: the vectors in this vector space are lists of \( n \) numbers, like \( (1, 2) \) or \( (4, 5, 6) \). These lists do not have an inherent orientation and may be written as convenient. A simple and pleasant example is with \( n = 2 \), namely \( \mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\} \), because it’s easy to draw vectors as arrows on the Cartesian plane.

7. For any set of vectors \( \{v_1, v_2, \ldots, v_k\} \) drawn from vector space \( V \), we say that this set is spanning if \( \text{Span}(v_1, v_2, \ldots, v_k) = V \). We know that \( \text{Span}(v_1, v_2, \ldots, v_k) \subseteq V \) holds for any set of vectors, so \( \{v_1, v_2, \ldots, v_k\} \) is spanning if \( \text{Span}(v_1, v_2, \ldots, v_k) \supseteq V \) also holds.

8. For any set of vectors \( \{v_1, v_2, \ldots, v_k\} \), their nondegenerate span is the set \( \{a_1v_1 + a_2v_2 + \cdots + a_kv_k\} \), where each of \( a_1, a_2, \ldots, a_k \) varies over every real number except \( a_1 = a_2 = \cdots = a_k = 0 \). Note that the regular span will always contain the vector 0, but the nondegenerate span may or may not contain 0.

9. For any set of vectors \( \{v_1, v_2, \ldots, v_k\} \), we say that this set is dependent if their nondegenerate span contains the vector 0. Otherwise, we say this set is independent; i.e. if their nondegenerate span does not contain the vector 0.

10. For any set of vectors \( \{v_1, v_2, \ldots, v_k\} \) drawn from vector space \( V \), we say that this set is a basis for \( V \) if it is both spanning and independent.
Comments on the Definitions:

1. Vector space closure in $V$ can be expressed in either of the following two ways:

   Closure 1: For every set of vectors $v_1, v_2, \ldots, v_k$ all in $V$, and for every set of real numbers $a_1, a_2, \ldots, a_k$, the linear combination $a_1v_1 + a_2v_2 + \cdots + a_kv_k$ is a vector again in $V$.

   Closure 2: Both (a) “scalar multiplication” and (b) “vector addition” hold, where:
   
   (a) For every vector $v$ in $V$, and every real number $a$, the product $av$ is a vector again in $V$.
   (b) For every two vectors $u, v$ in $V$, their sum $u + v$ is a vector again in $V$.

   Typically, if you already know that $V$ is closed, you use Closure 1. However, if you want to prove that $V$ is closed, you use Closure 2. There are other properties besides closure that must hold for $V$ to be a vector space; we will study these in detail later.

2. Every vector space contains a zero vector. This could be it; we call this the “trivial vector space”. If there is even one more vector, then there are infinitely many more; this can be proved by using scalar multiplication repeatedly.

3. If we set a linear function equal to a constant, e.g. $2x + 3y = 4$, we call this a linear equation.

4. The span is defined on (takes as input) a set of vectors, typically finite. Its product is (its value or output) also a set of vectors. This product is an infinite set, with the sole exception of $Span(0) = \{0\}$.

5. “Spanning”, “Dependent”, and “Basis” are all properties that a set of vectors does or does not possess.

6. The standard basis for the linear function space on a set of variables, is exactly that set of variables. For example, the standard basis for the linear function space on $\{x, y\}$ is $\{x, y\}$.

7. The standard basis for $P_n(t)$ is the set $\{1, t, t^2, \ldots, t^n\}$. Note that this contains not $n$ but $n+1$ vectors.

8. The standard basis for $P(t)$ is the set $\{1, t, t^2, \ldots\}$. Note that this contains infinitely many vectors.

9. The standard basis for $M_{m,n}$ is the set of $mn$ matrices, each of which has all zero entries except for a single 1 entry. The $mn$ possible locations of this 1 entry correspond to the different matrices. For example, $M_{2,2}$ has basis $\{(\begin{smallmatrix}1 & 0 \\ 0 & 0\end{smallmatrix}), (\begin{smallmatrix}0 & 1 \\ 0 & 0\end{smallmatrix}), (\begin{smallmatrix}0 & 0 \\ 1 & 0\end{smallmatrix})\}$. This is a set of $2 \times 2 = 4$ vectors.

10. The standard basis for $\mathbb{R}^n$ is denoted $\{e_1, e_2, \ldots, e_n\}$ where $e_i$ has all zeroes, except for a single 1 in the $i^{th}$ position. For example, if $n = 3$, $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$.

11. If a set of vectors $S$ contains two vectors, one of which is a multiple of the other, then $S$ is dependent. For example, $S = \{1 + 2t, 3 + 5t, 2 + 4t\}$ is dependent because the first vector is half of the last one. WARNING: the reverse need not hold. A set of vectors could be dependent even if no vector is a multiple of another. For example, $T = \{1 + 2t, 3 + 5t, 4 + 7t\}$ is dependent because the sum of the first two vectors, minus the third, equals 0.

12. An important theorem we will learn later is that all bases of a vector space have the same size. This size is called the “dimension” of the vector space. Hence you now know the dimension of our most important vector spaces. For example, $P_2(t)$ is three dimensional; all of its bases consist of three vectors.

13. If a subset of a vector space is closed, that subset must itself be a vector space. We call this a subspace of the original vector space. This allows us to construct lots of new vector spaces, as subspaces of the important vector spaces you already know.

Helpful Proof Techniques:

1. Know your definitions, as 100% of all proofs (not just in this course, but in all of mathematics) rely heavily on the precise statements of definitions.

2. In particular, know the difference between a scalar (number), a vector, a set of vectors, and a vector
space. If you’re working with something you need to always know which of these types it is.

3. To prove that a set of vectors $S$ is closed, let $u, v$ be arbitrary vectors in $S$, and $a$ be an arbitrary real number. You need to prove that $u + v$ and $au$ are both vectors in $S$.

4. To prove that a set of vectors $S$ is not closed, you need a single counterexample. Either find some $u, v \in S$ where $u + v \notin S$, or find some $u \in S$ and $a \in \mathbb{R}$ where $au \notin S$. Sometimes only one of these two approaches will work.

5. To prove that a set of vectors $S$ is spanning, take an arbitrary vector in $V$ and show how to express it as a linear combination of $S$.

6. To prove that a set of vectors $S$ is not spanning, you need a single counterexample. Select one vector in $V$ (it may be hard to find one that works), assume that it can be expressed as a linear combination of $S$, and derive a contradiction.

7. To prove that a set of vectors $S$ is dependent, you need to find a nondegenerate linear combination that gives the zero vector. This is typically harder the bigger $S$ is.

8. To prove that a set of vectors $S$ is independent, assume that a linear combination gives the zero vector, and prove that it must be the degenerate linear combination.

9. To prove that two sets are equal, prove that each is a subset of the other.

**Solved Problems**

1. Carefully state the definition of “Span”.
   The span of a set of vectors $\{v_1, v_2, \ldots, v_k\}$ is the set of all linear combinations $\{a_1v_1 + a_2v_2 + \cdots + a_kv_k\}$, where the $a_i$ each take on every real value.

2. Carefully state the definition of $P_3(t)$.
   $P_3(t)$ is the polynomial space in the variable $t$, of degree at most 3. Equivalently, this is $\{at^3 + bt^2 + ct + d\}$, where $a, b, c, d$ each take on every real value.

3. Carefully state the definition of “Dependent”.
   A set of vectors is dependent if their nondegenerate span contains the vector 0.

4. Carefully state the definition of $M_{2,2}$.
   $M_{2,2}$ is the matrix space consisting of all $2 \times 2$ matrices.

5. Carefully state the definition of “Basis”.
   A basis is a set of vectors that is both spanning and independent.

6. Give two vectors from the linear function space in $x$.
   Many examples are possible, such as $3x, -4x, \pi x, 0$.

7. Give two vectors from $\mathbb{R}^4$.
   Many examples are possible, such as $(0, 0, 0, 1), (1, 2, 3, 4), (-1, 0, 0, 2)$.

8. Consider the vector space $\mathbb{R}^3$, and set $v = (-3, 2, 0), u = (0, 1, 4)$. Calculate $2v - u$.
   $2v - u = 2(-3, 2, 0) - (0, 1, 4) = (-6, 4, 0) + (0, -1, -4) = (-6, 3, -4)$

9. Consider the vector space $M_{2,3}$, and set $u = (\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \end{pmatrix})$, $v = (\begin{pmatrix} 2 & 3 & 0 \\ 0 & 1 & 2 \end{pmatrix})$. Calculate $2v - u$.
   $2v - u = 2(\begin{pmatrix} 2 & 3 & 0 \\ 0 & 1 & 2 \end{pmatrix}) - (\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \end{pmatrix}) = (\begin{pmatrix} 3 & 4 & -1 \\ -1 & 3 & 4 \end{pmatrix})$.

10. Consider the vector space $P(t)$, and set $u = t + 1, v = t + 2$. Prove that $3t + 1$ is in $\text{Span}(u, v)$.
    Note that $5u - 2v = 5(t + 1) - 2(t + 2) = 3t + 1$, as desired. We find $5, -2$ by a side calculation; for example, $t = 2u - v$ and $1 = -u + v$ so $3t + 1 = 3(2u - v) + (-u + v) = 5u - 2v$. We will learn systematic ways to do this later.
11. Consider the vector space \( P(t) \), and set \( u = t + 1, v = t + 2 \). Prove that \( 3t^2 + 1 \) is not in \( \text{Span}(u, v) \).

Because \( u, v \) are both in \( P_1(t) \), their span is as well (in fact it is exactly \( P_1(t) \)). However \( 3t^2 + 1 \) is not in \( P_1(t) \).

12. Consider the linear function space in \( \{x, y, z\} \). Prove that \( \text{Span}(x, y, x - y) \).

Because \( x + y = 1x + 1y \) and \( x - y = 1x - 1y \), we conclude \( x + y, x - y \) are each in \( \text{Span}(x, y) \) and hence \( \text{Span}(x + y, x - y) \subseteq \text{Span}(x, y) \). On the other hand, \( x = \frac{1}{2}(x + y) + \frac{1}{2}(x - y) \) and \( y = \frac{1}{2}(x + y) - \frac{1}{2}(x - y) \), so \( x, y \) are each in \( \text{Span}(x + y, x - y) \) and hence \( \text{Span}(x, y) \subseteq \text{Span}(x + y, x - y) \).

13. Consider the set \( S \) of all \( v = (v_1, v_2) \) such that \( |v_1| \geq |v_2| \). This is a subset of \( \mathbb{R}^2 \). Is it closed?

For any scalar \( a \) and any vector \( v \) in \( S \), we calculate \( av = a(v_1, v_2) = (av_1, av_2) \). Because \( |v_1| \geq |v_2| \), we may multiply both sides by the nonnegative \( |a| \) to get \( |a||v_1| \geq |a||v_2| \) and hence \( |av_1| \geq |av_2| \). Hence \( av \) is a vector in \( S \); the first closure property holds.

We now take two vectors \( u, v \) in \( S \), and calculate \( u + v = (u_1, u_2) + (v_1, v_2) = (u_1 + v_1, u_2 + v_2) \). Must \( |u_1 + v_1| \geq |u_2 + v_2| \)? Perhaps not, so we need to find a specific counterexample. Many are possible, for example \( u = (3, 1), v = (-3, 1) \). Both of \( u, v \) are in \( S \), but \( u + v = (0, 2) \) is not. Hence the second closure property does NOT hold. Since both closure properties do not hold, \( S \) is not closed.

14. Consider vector space \( V \), and vectors \( v_1, v_2 \) in \( V \). Set \( S = \text{Span}(v_1, v_2) \). Prove that \( S \) is closed (and hence a subspace of \( V \)).

Let \( u, w \) be arbitrary vectors from \( \text{Span}(u, v) \). Then there are real numbers \( a_1, a_2, b_1, b_2 \) such that \( u = a_1v_1 + a_2v_2 \) and \( w = b_1v_1 + b_2v_2 \). We have \( u + w = a_1v_1 + a_2v_2 + b_1v_1 + b_2v_2 = (a_1 + b_1)v_1 + (a_2 + b_2)v_2 \), so \( u + w \) is in \( S \). This proves closure of vector addition. Let \( c \) be an arbitrary real number. Then \( cu = c(a_1v_1 + a_2v_2) = (ca_1)v_1 + (ca_2)v_2 \). Hence \( cu \) is in \( S \).

This proves closure of scalar multiplication.

In fact, a similar proof works not just for two vectors, but for any number.

15. Consider the vector space \( P_2(t) \), and set \( S = \{a_0 + a_1t + a_2t^2 : a_0 + a_1 + a_2 = 0\} \), a subset. Prove that \( S \) is closed.

Let \( u, v \) be arbitrary vectors in \( S \). Then there are real numbers \( a_0, a_1, a_2, b_0, b_1, b_2 \) such that \( u = a_0 + a_1t + a_2t^2 \) and \( v = b_0 + b_1t + b_2t^2 \), and also \( a_0 + a_1 + a_2 = 0 = b_0 + b_1 + b_2 \). We have \( u + v = (a_0 + b_0) + (a_1 + b_1)t + (a_2 + b_2)t^2 \), and \( (a_0 + b_0) + (a_1 + b_1) + (a_2 + b_2) = 0 \), so \( u + v \) is in \( S \). This proves closure of vector addition. Let \( c \) be an arbitrary real number. Then \( cu = (ca_0) + (ca_1)t + (ca_2)t^2 \). We have \( (ca_0) + (ca_1) + (ca_2) = c(a_0 + a_1 + a_2) = c0 = 0 \), so \( cu \) is in \( S \). This proves closure of scalar multiplication.

16. Consider the vector space \( P(t) \), and set \( u = t - 1, v = t^2 - 1, w = t^2 - t \). Prove that \( 3t + 1 \) is not in \( \text{Span}(u, v, w) \).

Method 1: Suppose \( 3t + 1 = a(t - 1) + b(t^2 - 1) + c(t^2 - t) = (b + c)t^2 + (a - c)t - (a + b) \). Equating coefficients of the polynomials in \( t \), we conclude that \( b + c = 0, a - c = 3, -a - b = 1 \). Adding these three equations we get \( 0 = 4 \); hence there is no solution.

Method 2: Let \( S = \{a_0 + a_1t + a_2t^2 : a_0 + a_1 + a_2 = 0\} \), a subset of \( P_2(t) \). \( S \) is closed by the preceding problem. Since \( u, v, w \in S \), also \( \text{Span}(u, v, w) \subseteq S \). However \( 3t + 1 \) is not in \( S \), so it cannot be in \( \text{Span}(u, v, w) \).

17. Consider the vector space \( \mathbb{R}^2 \), and set \( u = (1, 1), v = (2, 3), w = (0, 5) \). Prove that \( \{u, v, w\} \) is dependent.

To prove that \( \{u, v, w\} \) is dependent, we need to find a nondegenerate linear combination yielding zero. Consider \( 10u - 5v + w \), found by a side calculation. \( 10u - 5v + w = 10(1, 1) - 5(2, 3) + (0, 5) = (10, 10) - (10, 15) + (0, 5) = (0, 0) \). Hence, \( \{u, v, w\} \) is dependent.
18. Consider the vector space $\mathbb{R}^2$, and set $u = (2, 2), v = (3, 0)$. Prove that $\{u, v\}$ is independent.

To prove that $\{u, v\}$ is independent, we need to prove that any nondegenerate linear combination does not yield the zero vector. Suppose, to the contrary, that there were such a linear combination, i.e. some constants $a, b$ (not both zero) so that $au + bv = (0, 0)$. We calculate $au + bv = a(2, 2) + b(3, 0) = (2a, 2a) + (3b, 0) = (2a + 3b, 2a) = (0, 0)$. So, we must have $2a + 3b = 0$ and $2a = 0$. The second equation gives us $a = 0$; we plug that into the first equation and get $b = 0$. Hence, $a = b = 0$ and the linear combination was actually degenerate (a contradiction!). Hence $\{u, v\}$ is independent.

19. Consider the vector space $\mathbb{R}^3$, and set $u = (1, 1, 1), v = (-1, 0, 1), w = (1, 2, 3)$. Prove that $\{u, v, w\}$ is dependent.

To prove that $\{u, v, w\}$ is dependent requires a nondegenerate linear combination yielding the zero vector. We have $2u + v - w = 2(1, 1, 1) + 1(-1, 0, 1) - 1(1, 2, 3) = (2, 2, 2) + (-1, 0, 1) + (-1, -2, -3) = (0, 0, 0)$, so this set is dependent. To find this linear combination, we seek constants $a, b, c$ (not all zero) so that $au + bv + cw = (0, 0, 0)$. We calculate $au + bv + cw = (a, a, a) + (-b, 0, b) + (c, 2c, 3c) = (a - b + c, a + 2c, a + b + 3c) = (0, 0, 0)$.

Hence $a - b + c = 0, a + 2c = 0, a + b + 3c = 0$. This system has infinitely many solutions – choose $c$ arbitrarily, then $a = -2c, b = -c$. The example above corresponded to $c = -1$.

NOTE: No one of $u, v, w$ is a multiple of any one of the others, and yet they are dependent.

20. Consider the vector space $\mathbb{R}^2$, and set $u = (2, 3)$. Prove that $\{u\}$ is not spanning.

To prove that $\{u\}$ is not spanning, we must provide a counterexample. We claim that $(1, 1)$ cannot be expressed as a linear combination of $u$, because then for some $a$ we have $(1, 1) = a(2, 3) = (2a, 3a)$, and hence $2a = 1 = 3a$, which is impossible.

21. Consider the vector space $P_3(t)$. Prove that $\{t + 1, 2t - 1\}$ is spanning.

Consider an arbitrary vector in $P_3(t)$, say $at + b$. We consider the linear combination $\alpha(t + 1) + \beta(2t - 1)$, where $\alpha, \beta$ are real numbers given by $\alpha = \frac{a + 2b}{3}$ and $\beta = \frac{-a}{3}$ (found by a side calculation). We compute that $\alpha(t + 1) + \beta(2t - 1) = \frac{a + 2b}{3}(t + 1) + \frac{-a}{3}(2t - 1) = t(\frac{a + 2b}{3} + \frac{2a - b}{3}) + (\frac{a + 2b}{3} - \frac{a}{3}) = at + b$, as desired.

22. Consider the vector space $\mathbb{R}^2$, and set $u = (2, 2), v = (3, 0)$. Prove that $\{u, v\}$ is spanning.

To prove that $\{u, v\}$ is spanning, we need to prove that every vector can be expressed as a linear combination of $u, v$. Let $x = (x_1, x_2)$ be an arbitrary vector in $\mathbb{R}^2$. Set $a = x_2/2$ and set $b = (x_2 - x_1)/3$ (both real numbers no matter what $x$ is), found by a side calculation. We have $au + bv = a(2, 2) + b(3, 0) = (2a + 3b, 2a) = (x_1, x_2) = x$.

23. Consider the vector space $\mathbb{R}^2$, and set $u = (2, 2), v = (3, 0), w = (7, 5)$. Prove that $\{u, v, w\}$ is spanning.

To prove that $\{u, v, w\}$ is spanning, we need to prove that every vector can be expressed as a linear combination of $u, v, w$. Comparing with the previous problem, already every $x = au + bv + cw$, for some real $a, b$. Hence $x = au + bv + 0w$, a linear combination of $\{u, v\}$, so this set is also spanning.

24. Consider the vector space $\mathbb{R}^3$, and set $u = (1, 1, 1), v = (-1, 0, 1), w = (1, 2, 3)$. Prove that $\{u, v, w\}$ is not spanning.

To prove that $\{u, v, w\}$ is not spanning, we must find a counterexample. We claim that $x = (1, 1, 0)$ is such a counterexample (found by a tricky side calculation). Suppose we could express $x$ as a linear combination of $u, v, w$. Then, for some real constants $a, b, c$, we have $x = au + bv + cw = (a - b + c, a + 2c, a + b + 3c) = (1, 1, 0)$. Hence $a - b + c = 1, a + 2c = 1, a + b + 3c = 0$. Adding the first and third equations gives $2a + 4c = 1$, which is inconsistent with the second equation. Hence $x = (1, 1, 0)$ is not expressible as a linear combination of $\{u, v, w\}$, which is therefore not spanning.
25. Find two different bases for \( \mathbb{R}^2 \).

Many solutions are possible. An easy choice is the standard basis \( \{e_1, e_2\} = \{(1,0),(0,1)\} \).
An earlier problem showed that \( \{(2,2),(3,0)\} \) is spanning, and another proved that \( \{(2,2),(3,0)\} \) is independent; hence this set is a basis.

26. Consider the linear function space in \( \{x,y,z\} \). Set \( S = \text{Span}(x+y, x+z) \). Find two bases for \( S \).

A natural choice is \( \{x+y, x+z\} \); this set is spanning since \( \text{Span}(x+y, x+z) = S \) is exactly what we need. This set is independent because if \( a(x+y) + b(x+z) = 0 \) then \( a = b = 0 \) so no nondegenerate linear combination gives 0.
For another basis, consider \( \{x+y, -y+z\} \). These are both vectors from \( S \) since \( -y+z = -1(x+y) + 1(x+z) \). This set is independent because if \( a(x+y) + b(-y+z) = 0 \) then \( a = b = 0 \) again. To prove it is spanning it is enough to prove \( S \subseteq \text{Span}(x+y, -y+z) \). We have \( x+y = 1(x+y) + 0(-y+z) \), and \( x+z = 1(x+y) + 1(-y+z) \); hence the proof is complete.

**Supplementary Problems**

27. Carefully state the definition of “Vector Space”, and give ten examples.
28. Carefully state the definition of “Span”, and find a set of vectors whose span is itself.
29. Carefully state the definition of “Nondegenerate Span”, and give two examples.
30. Carefully state the definition of “\( M_{m,n} \)”, and give two vectors from \( M_{3,2} \).
31. Carefully state the definition of “Independent”, and give two examples from \( P_2(t) \).
32. Consider the vectors in \( \mathbb{R}^3 \) given by \( u = (1,2,3), v = (4,0,1), w = (-3,-2,5) \). Calculate \( 2u - 3v - 4w \).
33. Consider \( S \subseteq \mathbb{R}^2 \) of those vectors \( (v_1,v_2) \) such that \( 2v_1 + v_2 = 0 \). Determine if \( S \) is closed.
34. Consider \( S \subseteq \mathbb{R}^2 \) of those vectors \( (v_1,v_2) \) such that \( v_1v_2 = 0 \). Determine whether or not \( S \) is closed.
35. Consider the linear function space on \( \{x,y,z\} \). Determine whether or not \( x \in \text{Span}(x+y,x-z,y+z) \).
36. Consider the linear function space on \( \{x,y,z\} \). Determine whether or not \( x \in \text{Span}(x+y,x+z,y+z) \).
37. Consider the linear function space on \( \{x,y,z\} \). Determine whether or not \( x \in \text{Span}(x-y,x-z,y-z) \).
38. Consider \( S \subseteq M_{2,2} \) of those vectors \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) such that \( c = 0 \). Determine whether or not this is closed.
39. Consider \( S \subseteq M_{2,2} \) of those vectors \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) such that \( a + c = 0 \). Determine whether or not this is closed.
40. Consider \( S \subseteq M_{2,2} \) of those vectors \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) such that \( a + c = 1 \). Determine whether or not this is closed.
41. Consider the vector space \( \mathbb{R}^2 \), and set \( u = (2,6), v = (-3,-9) \). Determine if \( \{u,v\} \) is independent.
42. Consider \( \mathbb{R}^2 \), and set \( u = (2,6), v = (-3,-9), w = (5,15) \). Determine if \( \{u,v,w\} \) is independent.
43. Consider the vector space \( \mathbb{R}^2 \), and set \( u = (2,6), v = (0,-9) \). Determine if \( \{u,v\} \) is independent.
44. Consider the vector space \( P_2(t) \). Determine whether or not \( \{1,2t\} \) is independent.
45. Consider the vector space \( P_2(t) \). Determine whether or not \( \{0,1,2t\} \) is spanning.
46. Consider the vector space \( P_3(t) \). Determine whether or not \( \{6t+2,-9t-3\} \) is spanning.
47. Consider the vector space \( M_{2,2} \). Determine if \( \left\{ \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\} \) is spanning.
48. Consider the vector space \( M_{2,2} \). Determine if \( \left\{ \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\} \) is spanning.
49. Consider the vector space \( M_{2,2} \). Determine if \( \left\{ \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \right\} \) is spanning.
50. Which of the sets given in problems 41-49 are bases of their respective vector spaces?

Answers to Supplementary Problems: (WARNING: these are just answers, NOT thoroughly justified solutions)