Math 254 Fall 2014 Exam 4 Solutions

1. Carefully state the definition of "span". What type of object is it? Choose one from: [vector, scalar, vector space, list of numbers, set of numbers, linear combination, other]

   The span of a set of vectors is the set of all linear combinations of those vectors. This is a *vector space*.

2. Carefully state five of the eight vector space axioms.

   These are listed on p.152 of the text. To receive full credit, you need to include complete statements, such as "For all vectors u, v, $u+v=v+u$." "$u+v=v+u$" is incomplete.

   The remaining three problems concern the vector space $V=\mathbb{R}^4$ and the subsets $S=\{(a,b,c,d): a+b=c+d=0\}$, $T=\{(a,b,c,d): a+c=d=0\}$.

3. Prove that S is a subspace of V.

   We need to prove closure. Suppose $u=(a,b,c,d), v=(a',b',c',d')$ are each in S. Then $a+b=c+d=0$ and $a'+b'=c'+d'=0$, and by adding we get $(a+a')+(b+b')=(c+c')+(d+d')=0$; hence $(a+a',b+b',c+c',d+d')=u+v$ is in S. Multiplying by k we get $ka+kb=kc+kd=k0=0$, so $(ka,kb,kc,kd)=ku$ is in S.

4. Prove that $S\cap T=\{0\}$. (you may assume that S,T are vector spaces)

   Let $x=\left[\begin{smallmatrix} a & b & c & d\end{smallmatrix}\right]^T$. S is the solution space to $\left[\begin{smallmatrix}1&1&0&0\\0&0&1&1\end{smallmatrix}\right]x=0$, while T is the solution space to $\left[\begin{smallmatrix}1&0&1&0\\0&0&0&1\end{smallmatrix}\right]x=0$. Hence $S\cap T$ is the solution space to $\left[\begin{smallmatrix}1&1&0&0\\0&0&1&1\\1&0&1&0\\0&0&0&1\end{smallmatrix}\right]x=0$. This matrix has row canonical form I_4; hence the only solution is $x=0$.

5. Prove that each of the four standard basis vectors e_1,e_2,e_3,e_4 are in $S+T$.

   We must express each as the sum of a vector from S and a vector from T:
   $e_1=(1,-1,0,0)+(0,1,0,0)$ $\quad e_2=(0,0,0,0)+(0,1,0,0)$
   $e_3=(1,-1,0,0)+(-1,1,1,0)$ $\quad e_4=(1,-1,-1,1)+(-1,1,1,0)$

   Together with (4) this proves that $V=S\oplus T$. Later, when we learn the dimension theorem, we will have $dim(S+T)+dim(S\cap T)=dim(S)+dim(T)$, which will allow us to calculate $dim(S+T)=4$ and hence $S+T=V$ directly.