Math 254 Fall 2014 Exam 3 Solutions

1. Carefully state the definition of "spanning". Give a spanning set for $M_{2,3}$.

A set of vectors is spanning if their span is the entire vector space. A correct answer to the second part must consist of a set of at least six 2×3 matrices, such as
$\{\left(\begin{smallmatrix}1&0&0\\0&0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&1&0\\0&0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&1\\0&0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\1&0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\0&1&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\0&0&1\end{smallmatrix}\right)\}$

2. Prove or disprove the following statement:
Every skew-symmetric 2×2 matrix must have all of its diagonal entries equal to 0.

The statement is true. Proof: if $A = [a_{i,j}]$ is skew-symmetric, then $a_{i,j} = -a_{j,i}$ for each i and each j. In particular, $a_{1,1} = -a_{1,1}$ and $a_{2,2} = -a_{2,2}$. The only real number equal to its own negative is zero.

The remaining three problems concern the matrix $A = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix}$.

3. Find a symmetric matrix B and a skew-symmetric matrix C such that $A = B + C$.

Take $B = \frac{1}{2}(A + A^T) = \begin{bmatrix} \frac{1}{3} & 0 & \frac{2}{3} \\ 0 & -\frac{1}{3} & 0 \\ \frac{2}{3} & 0 & \frac{1}{3} \end{bmatrix}$ and $C = \frac{1}{2}(A - A^T) = \begin{bmatrix} 0 & -\frac{2}{3} & 0 \\ \frac{2}{3} & 0 & -\frac{2}{3} \\ 0 & \frac{2}{3} & 0 \end{bmatrix}$.

4. Calculate A^{-1}. Be sure to justify each step.

$$[A|I] = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} & 1 & 0 & 0 \\ \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} & 0 & 1 & 0 \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 2 & 3 & 0 & 0 \\ 2 & -1 & -2 & 0 & 3 & 0 \\ 2 & 2 & 1 & 0 & 0 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 2 & 3 & 0 & 0 \\ 0 & 3 & -6 & -6 & 3 & 0 \\ 0 & 6 & -3 & -6 & 0 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 & -1 & 2 & 0 \\ 0 & 3 & -6 & -6 & 3 & 0 \\ 0 & 0 & 9 & 6 & -6 & 3 \end{bmatrix} \to$$
$$\begin{bmatrix} 1 & 0 & 0 & \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ 0 & 3 & 0 & -2 & -1 & 2 \\ 0 & 0 & 9 & 6 & -6 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ 0 & 1 & 0 & -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 1 & \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix} = [I|A^{-1}]. \text{ Hence } A^{-1} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}.$$

Step 1: $3R_1 \to R_1, 3R_2 \to R_2, 3R_3 \to R_3$
Step 2: $-2R_1 + R_2 \to R_2, -2R_1 + R_3 \to R_3$
Step 3: $(2/3)R_2 + R_1 \to R_1, -2R_2 + R_3 \to R_3$
Step 4: $(2/9)R_3 + R_1 \to R_1, (2/3)R_3 + R_2 \to R_2$
Step 5: $(1/3)R_2 \to R_2, (1/9)R_3 \to R_3$.

5. Determine whether or not A is orthogonal. Be sure to justify your answer.

Looking at A^{-1}, calculated in (4), we see that it equals A^T. Hence A is orthogonal.

ALTERNATE SOLUTION: Compute AA^T, and check that it equals I_3.