Math 254 Fall 2014 Exam 2b Solutions

1. Carefully state the definition of the standard vector space $\mathbb{R}^n$. Give an independent set of two vectors, drawn from $\mathbb{R}^3$.

$\mathbb{R}^n$ is the set of all n-tuples of real numbers. Correct answers to the second part must be *sets* containing two 3-tuples, such as $\{(1,2,3),(1,1,1)\}$ or $\{(1,0,0),(0,1,0)\}$.

2. Prove or disprove the following statement:
For all 2×2 matrices A, B, each in echelon form, their sum $A+B$ must be in echelon form.

The statement is false; to disprove it we need one specific counterexample. This consists of A, B that *are* both in echelon form, but their sum $A+B$ is *NOT* in echelon form. One counterexample is $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}, B = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}, A+B = \begin{bmatrix} 0 & 3 \\ 0 & 4 \end{bmatrix}$.

The remaining three problems concern the matrix $A = \begin{bmatrix} 1 & 3 & -3 & 0 & -4 \\ 0 & 0 & 1 & 3 & -1 \\ 3 & 9 & -2 & 5 & -3 \\ 2 & 6 & 3 & -1 & 11 \\ 5 & 15 & 0 & -7 & 17 \end{bmatrix}$.

3. Place A in echelon form. Be sure to justify each step.

$$A \to \begin{bmatrix} 1 & 3 & -3 & 0 & -4 \\ 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & 7 & 5 & 9 \\ 0 & 0 & 9 & -1 & 19 \\ 0 & 0 & 15 & -7 & 37 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & -3 & 0 & -4 \\ 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & -16 & 16 \\ 0 & 0 & 0 & -28 & 28 \\ 0 & 0 & 0 & -52 & 52 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & -3 & 0 & -4 \\ 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & -16 & 16 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ (in echelon form)}$$

Step 1: $R_3 - 3R_1 \to R_3, R_4 - 2R_1 \to R_4, R_5 - 5R_1 \to R_5$
Step 2: $R_3 - 7R_2 \to R_3, R_4 - 9R_2 \to R_4, R_5 - 15R_2 \to R_5$
Step 3: $R_4 - \frac{28}{16}R_3 \to R_4, R_5 - \frac{52}{16}R_3 \to R_5$

4. Place A in row canonical form. Be sure to justify each step. You should begin with your answer from (3).

$$(3) \to \begin{bmatrix} 1 & 3 & -3 & 0 & -4 \\ 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & -3 & 0 & -4 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ (in row canonical form)}$$

Step 1: $\frac{-1}{16}R_3 \to R_3$ Step 2: $-3R_3 + R_2 \to R_2$ Step 3: $3R_2 + R_1 \to R_1$

5. Write down a linear system for which A is an augmented matrix, and interpret your answer from (4) to write down the general solution for your system.

A possible system: $\{1a+3b-3c+0d=-4, 0a+0b+1c+3d=-1, 3a+9b-2c+5d=-3, 2a+6b+3c-1d=11, 5a+15b+0c-7d=17\}$. This has general solution $\{(2-3t,t,2,-1) : t\in\mathbb{R}\}$, or $\{(2-3b,b,2,-1) : b\in\mathbb{R}\}$. Note there is one free variable and three bound variables.