Math 254 Fall 2014 Exam 12 Solutions

1. Carefully state the definition of "basis". Give a basis for $M_{2,2}$.

A basis is a set of vectors that is independent and spanning. $\{\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)\}$

2. Give an example of a quadratic form $q(x, y)$ such that there are two vectors $u = (x_u, y_u)$, $v = (x_v, y_v)$ with $q(u) = 0$ and $q(v) = 0$ but $q(u + v) \neq 0$.

Solution 1: This is exercise 12.35 in the book, and the solution given is $q(x, y) = x^2 - y^2$, with $u = (1, 1)$, $v = (1, -1)$. Note that $u + v = (2, 0)$ and $q(2, 0) = 4 - 0 = 4$.

Solution 2: We can find one systematically as $q'(x, y) = x^2 + y^2 + kxy$. If we want $u = (2, -1), v = (-1, 2)$, then $4 + 1 - 2k = 0$ and $1 + 4 - 2k = 0$, so $k = 2.5$ and $q'(x, y) = x^2 + y^2 + 2.5xy$. Now $u + v = (1, 1)$ and $q'(1, 1) = 1 + 1 + 2.5 = 4.5$.

The remaining problems all concern $A = \left(\begin{smallmatrix}1&2&3\\2&1&6\\3&6&9\end{smallmatrix}\right)$.

3. Find invertible matrix P and diagonal matrix D such that $D = P^T AP$.

$\left(\begin{smallmatrix}1&2&3&1&0&0\\2&1&6&0&1&0\\3&6&9&0&0&1\end{smallmatrix}\right) \to \left(\begin{smallmatrix}1&0&3&1&0&0\\0&-3&0&-2&1&0\\3&0&9&0&0&1\end{smallmatrix}\right)$, via $R_2 - 2R_1 \to R_2$ and $C_2 - 2C_1 \to C_2$.
$\left(\begin{smallmatrix}1&0&3&1&0&0\\0&-3&0&-2&1&0\\3&0&9&0&0&1\end{smallmatrix}\right) \to \left(\begin{smallmatrix}1&0&0&1&0&0\\0&-3&0&-2&1&0\\0&0&0&-3&0&1\end{smallmatrix}\right)$, via $R_3 - 3R_1 \to R_3$ and $C_3 - 3C_1 \to C_3$.
Hence $D = \left(\begin{smallmatrix}1&0&0\\0&-3&0\\0&0&0\end{smallmatrix}\right)$, $P^T = \left(\begin{smallmatrix}1&0&0\\-2&1&0\\-3&0&1\end{smallmatrix}\right)$, and $P = \left(\begin{smallmatrix}1&-2&-3\\0&1&0\\0&0&1\end{smallmatrix}\right)$.

4. Use D to find the rank and signature of A. Is A positive definite?

We have $n_+ = 1$ (**p** in the book's notation), $n_- = 1$ (**n** in the book's notation), and $n_0 = 1$. $rank(A) = n_+ + n_i = 2$, $sig(A) = n_+ - n_- = 0$. For A to be positive definite, $n_- = n_0 = 0$ would have to hold (equivalently, $n_+ = 3$ would have to hold). Since it doesn't, A is *not* positive definite.

5. Consider the quadratic form $q(x, y, z) = x^2 + y^2 + 9z^2 + 4xy + 6xz + 12yz = 1$, and the surface given by $q(x, y, z) = 1$. Use A, P, D to diagonalize this quadratic form. Write the surface in the new variables, and show the relationship between the old and new variables. What is the name of this surface?

Note that $q(x, y, z) = \left(\begin{smallmatrix}x\\y\\z\end{smallmatrix}\right)^T A \left(\begin{smallmatrix}x\\y\\z\end{smallmatrix}\right)$. Under the change of variables $P\left(\begin{smallmatrix}r\\s\\t\end{smallmatrix}\right) = \left(\begin{smallmatrix}x\\y\\z\end{smallmatrix}\right)$, we have the diagonalized form $q(r, s, t) = \left(\begin{smallmatrix}r\\s\\t\end{smallmatrix}\right)^T P^T AP \left(\begin{smallmatrix}r\\s\\t\end{smallmatrix}\right) = \left(\begin{smallmatrix}r\\s\\t\end{smallmatrix}\right)^T D \left(\begin{smallmatrix}r\\s\\t\end{smallmatrix}\right) = r^2 - 3s^2 + 0t^2$. The surface becomes $r^2 - 3s^2 + 0t^2 = 1$, which is called a hyperbolic cylinder.