Name:

Math 254 Fall 2013 Final Exam

Please read the following directions:
Please print your name in the space provided, using large letters, as "First LAST". Books, notes, calculators, and other aids are not permitted on this exam, apart from a single 3"×5" note card. Please write legibly, with plenty of white space. Please put your answers in the designated areas. Show all necessary work in your solutions; if you are unsure, show it. Cross out work you do not wish graded; incorrect work can lower your grade. All problems are worth 5-10 points; your total will be scaled to the standard 100 point scale. You have approximately 120 minutes. The back of the exam may be used for scratch paper, if necessary.

1. Carefully state the definition of "dimension", as it applies to vector spaces. Give two examples, a four-dimensional one and an infinite-dimensional one.

2. Carefully state the definition of "independent". Give two examples from $M_{2,2}$.

3. Carefully state the definition of "subspace". Give two examples from within $\mathbb{R}^3$.

4. Use Gaussian Elimination to solve the linear system $\{2x + 3y = 1, 3x + 4y = 2\}$.

5. Use Cramer's Rule to solve the linear system $\{2x + 3y = 1, 3x + 4y = 2\}$.

6. In vector space $P_2(t)$, set $p_1(t) = 3 - t^2, p_2(t) = 3 + 2t + t^2, p_3(t) = t + t^2$. Find a basis for $Span(\{p_1, p_2, p_3\})$, and its dimension.

Problems 7-10 all concern the vector space $P_2(t)$ and the function $T : P_2(t) \to P_2(t)$ given by $T(f(t)) = t^2 f(t^{-1})$. For example, $T(2 + 3t + 4t^2) = t^2(2 + 3t^{-1} + 4t^{-2}) = 4 + 3t + 2t^2$.

7. Prove that T is a linear transformation.

8. Find representation $[T]_E$, for standard basis $E = \{1, t, t^2\}$.

9. Find representation $[T]_S$, for basis $S = \{1 + t^2, t + t^2, t^2\}$.

10. With T as in problems 7-9, find the characteristic polynomial of T and its eigenvalues.

Problems 11-12 both concern the vector space $M_{2,2}$, the standard inner product given by $\langle A, B\rangle = tr(B^T A)$, and vectors $X = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $Y = \left(\begin{smallmatrix} 2 & 1 \\ 0 & 1 \end{smallmatrix}\right)$.

11. Find the angle between X and Y.

12. Find an orthogonal basis for $Span(\{X, Y\})$.

13. Suppose that $A, B \in M_{n,n}$. Prove that if A is not invertible, then AB is not invertible.

14. Suppose that U, W are both subspaces of V. Prove that $U \cap W$ is a subspace of V.

15. Prove or find a counterexample for the following statement: If A, B are similar square matrices, then $Ax = 0$ and $Bx = 0$ have the same set of solutions.