# Math 254 Fall 2013 Exam 8 Solutions

1. Carefully state the definition of "vector space". Give two three-dimensional examples.

A vector space is a collection of objects (called vectors), a set of scalars (typically $\mathbb{R}$), and a way to add vectors and multiply vectors by scalars. Two familiar three-dimensional examples are $\mathbb{R}^3$ and $P_2(t)$.

2. Carefully state the definition of "linear transformation". Give two examples on $P_2(t)$.

A linear transformation is a function $f$ from a vector space $U$ to a vector space $V$, satisfying: (1) For all $u, v \in U$, $f(u+v) = f(u) + f(v)$, and (2) For all $u \in U$ and $k \in \mathbb{R}$, $f(ku) = kf(u)$. Many examples are possible such as $f(p(t)) = p(t)$ (identity), $f(p(t)) = -p(t)$, $f(at^2+bt+c) = bt^2 + (a+c)t + a$.

3. Consider the mapping $f : P_1(t) \to \mathbb{R}^3$ given by $f(a+bt) = (a, a+b, 2b)$. Determine, with justification, whether or not $f$ is linear.

1. Let $a+bt, a'+b't$ be two arbitrary vectors in $P_1(t)$. We have $f(a+bt) + f(a'+b't) = (a, a+b, 2b)+(a', a'+b', 2b') = (a+a', a+b+a'+b', 2b+2b') = (a+a', (a+a')+(b+b'), 2(b+b')) = f((a+a') + (b+b')t) = f((a+bt) + (a'+b't))$. This is the first required property.
2. Let $a+bt$ be arbitrary in $P_1(t)$, and let $k \in \mathbb{R}$. We have $kf(a+bt) = k(a, a+b, 2b) = (ka, ka+kb, 2kb) = f(ka+kbt)$. This is the second required property, so the answer is YES.

4. Consider the linear mapping $g : \mathbb{R}^3 \to P_2(t)$ given by $g((a, b, c)) = a + (b+c)t + at^2$. Find a basis for the kernel of $g$, and find a basis for the image of $g$.

If $(a, b, c)$ is in the kernel of $g$, then $g((a, b, c)) = a+(b+c)t+at^2 = 0$, so $a = 0, b+c = 0, a = 0$. This is a one-dimensional space, with basis $\{(0, 1, -1)\}$.

By the rank-nullity theorem, $dim(Im\ g)+dim(Ker\ g) = dim(\mathbb{R}^3)$, so $dim(Img) = 2$ and any basis for $Im\ g$ will consist of two (linearly independent) vectors. One example is $\{1+t^2, t\}$.

5. Let $f, g$ be as in problems 3,4. Consider the linear mapping $h : P_1(t) \to P_2(t)$ given by $h = g \circ f$. Calculate $h(1+2t)$, and determine (with justification) whether $h$ is an isomorphism.

We have $h(1+2t) = (g \circ f)(1+2t) = g(f(1+2t)) = g(1, 3, 4) = 1 + 7t + t^2$. The linear map $h$ is NOT an isomorphism, and here are two possible explanations why not:

1. We calculated in Problem 4 that $dim(Im\ g) = 2$, so $dim(Im\ h) \leq 2$. But $dim(P_2(t)) = 3$, so $h$ cannot be onto.
2. By the rank-nullity theorem $dim(P_1(t)) = dim(Ker\ h)+dim(Im\ h)$. Because $dim(P_1(t)) = 2$ and $dim(Ker\ h) \geq 0$, we must have $dim(Im\ h) \leq 2 < dim(P_2(t))$, so $h$ cannot be onto.

Extra: Consider the linear mapping $f : M_{2,2} \to M_{2,2}$ given by $f(A) = \frac{1}{2}(A + A^T)$. Find a basis for the kernel of $f$, and find a basis for the image of $f$. Are either of these spaces familiar?

Suppose $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is in the kernel of $f$. Then $0 = f(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \begin{pmatrix} a & (b+c)/2 \\ (b+c)/2 & d \end{pmatrix}$, so $a = 0 = d$ and $\frac{1}{2}(b+c) = 0$. This is a one-dimensional subspace with basis $\{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\}$, also known as the set of all skew-symmetric $2 \times 2$ matrices. By the rank-nullity theorem, $dim(Im\ f) = 3$; by applying $f$ to each of $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, we find a basis for $Im\ f$ of $\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\}$. This subspace is also known as the set of all symmetric $2 \times 2$ matrices.