Math 254 Fall 2013 Exam 5 Solutions

1. Carefully state the definition of "polynomial space". Give a set of three vectors from $P_1(t)$.

The polynomial space in a variable is the vector space consisting of all polynomials in that single variable. A set of three vectors from $P_1(t)$ is $\{0, 1+t, 2-3t\}$.

2. Let S consist of all skew-symmetric 3×3 matrices. S is a 3-dimensional vector space; find a basis for S.

A skew-symmetric matrix A satisfies $A^T = -A$. Take $\left\{ \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \right\} = T$. Since we are told the dimension is 3, and T has 3 vectors, we may either prove that T is independent or spanning. Independent is easier, as we don't need to determine the structure of S. Suppose a linear combination of them equals zero. Hence $a\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = 0$. Then $\begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} = \overline{0}$, so $a=b=c=0$, so this was a degenerate linear combination.

The remainder concerns matrix $A = \begin{pmatrix} 2 & 1 & -3 & 5 \\ 4 & 2 & 0 & 3 \\ 2 & 1 & 3 & -2 \end{pmatrix}$ which has echelon form $\begin{pmatrix} 4 & 2 & 0 & 3 \\ 0 & 0 & 6 & -7 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

3. Let $S = rowspace(A)$. Determine the dimension of S and find a basis.

Because the echelon form of A has two nonzero rows, $dim(S) = 2$. Its basis is those rows, namely $\{(4,2,0,3), (0,0,6,-7)\}$.

4. Let $T = columnspace(A)$. Determine the dimension of T and find a basis.

Because the echelon form of A has two pivots, $dim(T) = 2$. Its basis corresponds to those columns of A that end up with pivots, namely the first and third ones, so $\{(2,4,2), (-3,0,3)\}$.

5. Set $U = Span\{(2,4,2),(1,2,1)\}, V = Span\{(-3,0,3),(5,3,-2)\}$, subspaces of $\mathbb{R}^3$. Find $dim(U), dim(V), dim(U+V)$, and $dim(U\cap V)$.

$dim(U) = 1$ since the first vector is twice the second. $dim(V) = 2$ since the first vector isn't a multiple of the second. [Warning: this procedure only works to test independence for 2 vectors, not more.] $U+V$ turns out to be exactly $columnspace(A)$, so $dim(U+V) = 2$ by Problem 4. We have the lovely theorem telling us that $dim(U) + dim(V) = dim(U+V) + dim(U\cap V)$, so $1+2 = 2 + dim(U\cap V)$, and hence $dim(U\cap V) = 1$.

Extra: Consider the function space $F(x)$. Let $S = Span\{1, \sin x, \cos x, \sin^2 x, \cos^2 x\}$, a subspace of $F(x)$. Find a basis for S.

Because $\sin^2 x + \cos^2 x = 1$, $\cos^2 x$ isn't needed; we will show that $\{1, \sin x, \cos x, \sin^2 x\}$ is a basis for S. It suffices to prove that this set is independent, because the dimension of $Span\{1, \sin x, \cos x, \sin^2 x, \cos^2 x\} = Span\{1, \sin x, \cos x, \sin^2 x\} \leq 4$. Suppose we had a linear combination $a + b\sin x + c\cos x + d\sin^2 x = 0$. This would have to hold for *all* x. Taking $x = 0$ gives $a + c = 0$, while taking $x = \pi$ gives $a - c = 0$. Hence $a = c = 0$. Taking $x = \pi/2$ gives $b + d = 0$, while taking $x = 3\pi/2$ gives $-b + d = 0$. Hence $b = d = 0$, so the linear combination was degenerate, and hence this set is independent.