Math 254 Fall 2013 Exam 4 Solutions

1. Carefully state the definition of "nondegenerate span". Give an example from $P_2(t)$ whose nondegenerate span contains $\overline{0}$.

   The nondegenerate span of a set of vectors is the set of all linear combinations of those vectors, *except* the all-zero linear combination. Many examples are possible, such as $\{0\}$ or $\{t, 2t\}$.

2. Carefully state five of the eight vector space axioms.

   These are listed on p.152 of the text. To receive full credit, you need to include complete statements, such as "For all vectors u, v, $u + v = v + u$." "$u + v = v + u$" is incomplete.

3. Let V be the set of all 2×2 symmetric matrices. Prove that V is a vector space.

   As a subset of $M_{2,2}$, all eight axioms are inherited and we only need prove closure. Let $M, N \in V$. Then $M = M^T, N = N^T$. Now, $(M+N)^T = M^T + N^T = M+N$, so $(M+N) \in V$, which proves closure under vector addition. Let $k \in \mathbb{R}$. Then $(kM)^T = k(M^T) = kM$. Hence $kM \in V$, which proves closure under scalar multiplication.

4. Within $\mathbb{R}^2$, let $u = (1, 2), v = (2, 3)$, and $w = (3, 4)$. Determine, with justification, whether or not $w \in Span(u, v)$.

   We begin by putting u, v as rows of a matrix, then finding its row canonical form, using $-2R_1 + R_2 \to R_2, 2R_2 + R_1 \to R_1, -R_2 \to R_2$, which gives $\left(\begin{smallmatrix} 1 & 2 \\ 2 & 3 \end{smallmatrix}\right) \to \left(\begin{smallmatrix} 1 & 2 \\ 0 & -1 \end{smallmatrix}\right) \to \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \to \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

   Solution 1: We now put u, v, w as rows of a matrix, then finding its row canonical form. We begin the same way as before, then $-3R_1 + R_3 \to R_3, -4R_2 + R_3 \to R_3$, which gives $\left(\begin{smallmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{smallmatrix}\right) \to \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \\ 3 & 4 \end{smallmatrix}\right) \to \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 4 \end{smallmatrix}\right) \to \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. Since this matrix and the first are both row canonical and have the same nonzero rows, they have the same rowspace, so $w \in Span(u, v)$.

   Solution 2: Because $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ has two-dimensional rowspace, and $\mathbb{R}^2$ is two dimensional, they must coincide. Hence $\{u, v\}$ is a basis, and *every* vector in $\mathbb{R}^2$ is in $Span(u, v)$, including w.

5. Within $P_2(t)$, let $U = Span(1 + t, 1 + t^2)$, $V = Span(1)$. Prove that $P_2(t) = U \oplus V$.

   We first prove that $P_2(t) = U + V$. arbitrary polynomial $at^2 + bt + c$ may be expressed as the sum of a polynomial from U, and one from V. We have $at^2 + bt + c = u + v$, where $u = a(1 + t^2) + b(1 + t)$ and $v = (c - a - b)1$. Note that $u \in U$, $v \in V$.

   Solution 1: We note that u, v are unique because a, b determine the element of U, and then the element of V is determined uniquely. Hence $P_2(T) = U \oplus V$.

   Solution 2: We prove that $U \cap V = \{0\}$. Suppose that $\alpha(1 + t) + \beta(1 + t^2) = \gamma(1)$ is in $U \cap V$. Since $\gamma(1)$ has no t, t^2 terms, then $\alpha = \beta = 0$ and this vector is actually $\overline{0}$.

Extra: Let V be the set of all 2×2 orthogonal matrices. Determine whether or not V is a vector space, and justify your answer.

The answer is no. V is not closed under either scalar multiplication or vector addition. One example suffices, but I'll give two (one for each). First, $I \in V$ since $I(I)^T = I^2 = I$, but $2I \notin V$ since $(2I)(2I)^T = 4I \neq I$. Second, $I \in V$, but $I + I = 2I \notin V$ by the same calculation.