Math 254 Fall 2013 Exam 2b Solutions

1. Carefully state the definition of "span". Describe $Span(1, t, t^2)$.

   The *span* of a set of vectors S is the set of all vectors obtained by all linear combinations of S. Alternatively, the *span* of a set of vectors $\{v_1, v_2, \ldots, v_k\}$ is the set $\{\sum_{i=1}^{k} a_i v_i : a_i \in \mathbb{R}\}$. The set $Span(1, t, t^2)$ is the set of all polynomials of degree at most two, also known as $P_2(t)$.

2. Carefully state the definition of "associated homogeneous linear system". Give an example, and explain its purpose.

   A linear system has an associated homogeneous linear system, found by replacing all the constants by 0 while leaving the coefficients of the variables unchanged. Its purpose is to find the general solution to the original system using the general solution to the associated homogeneous system (and a particular solution to the original system). An example is $\{x + y = 1, x + 2y = 3\}$ has associated homogeneous system $\{x + y = 0, x + 2y = 0\}$.

The remaining three problems all concern the matrix $A = \begin{bmatrix} 2 & 2 & 3 & 3 & 4 \\ 2 & 3 & 4 & 1 & 4 \\ 2 & 4 & 5 & 0 & 5 \\ 2 & 3 & 4 & 3 & 6 \end{bmatrix}$.

3. Place A in echelon form. Be sure to justify each step.

   Step 1: $R_2 - R_1 \to R_2, R_3 - R_1 \to R_3, R_4 - R_1 \to R_4$. This results in $\begin{bmatrix} 2 & 2 & 3 & 3 & 4 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 2 & 2 & -3 & 1 \\ 0 & 1 & 1 & 0 & 2 \end{bmatrix}$.

   Step 2: $R_3 - 2R_2 \to R_3, R_4 - R_2 \to R_4$. This results in $\begin{bmatrix} 2 & 2 & 3 & 3 & 4 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 & 2 \end{bmatrix}$.

   Step 3: $R_4 - 2R_3 \to R_4$. This results in $\begin{bmatrix} 2 & 2 & 3 & 3 & 4 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

   This last matrix is in echelon form.

4. Place A in row canonical form. Be sure to justify each step. You should begin with your answer from (3).

   Step 1: $R_1 - 2R_2 \to R_1$. This results in $\begin{bmatrix} 2 & 0 & 1 & 7 & 4 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

   Step 2: $R_1 - 7R_3 \to R_1, R_2 + 2R_3 \to R_2$. This results in $\begin{bmatrix} 2 & 0 & 1 & 0 & -3 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

   Step 3: $\frac{1}{2}R_1 \to R_1$. This results in $\begin{bmatrix} 1 & 0 & \frac{1}{2} & 0 & -\frac{3}{2} \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

   This last matrix is in row canonical form.

5. Write down a linear system for which A is an augmented matrix, and interpret your answer from (4) to write down the general solution for your system.

   We need to choose variables. I pick a, b, c, d; then the system becomes $\{2a+2b+3c+3d = 4, 2a+3b+4c+d = 4, 2a+4b+5c = 5, 2a+3b+4C+3d = 6\}$.

   We use back-substitution on the row canonical matrix, to first get $d = 1$. Then, c is free and $b = 2 - c$. Lastly, $a = -\frac{3}{2} - \frac{1}{2}c$. Putting it all together, the general solution is $\{(\frac{-3-c}{2}, 2 - c, c, 1) : c \in \mathbb{R}\}$.

Extra: Consider the linear system in x, y, z given by $\{x + 2y + 2z = 1, y + z = 2, x + y + tz = -1\}$. For which values of t (if any) are there no solutions, one solution, infinitely many solutions? Find all solutions.

$\begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ 1 & 1 & t & -1 \end{bmatrix}$ has echelon form $\begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & t-1 & 0 \end{bmatrix}$. If $t \neq 1$, then there is one solution, $(-3, 2, 0)$. If $t = 1$, there are infinitely many solutions $\{(-3, 2 - z, z) : z \in \mathbb{R}\}$.