Math 254 Fall 2013 Exam 12 Solutions

1. Carefully state the definition of "matrix space" $M_{m,n}$. Give two six-dimensional examples.

Matrix space $M_{m,n}$ is the vector space consisting of all $m \times n$ matrices. Exactly four of these are six-dimensional: $M_{1,6}, M_{2,3}, M_{3,2}, M_{6,1}$.

2. Give a 4×4 matrix, in Jordan canonical form, whose minimal polynomial is $(t-2)^3$.

Because $\lambda = 2$ is the only root of the minimal polynomial, it is the only eigenvalue of the matrix. Because it appears with multiplicity 3, the largest Jordan block for $\lambda = 2$ is of size 3. There is only room for one more block, which must be of size 1. Hence there are exactly two possible answers: $\begin{pmatrix} 2&0&0&0\\0&2&1&0\\0&0&2&1\\0&0&0&2 \end{pmatrix}$ and $\begin{pmatrix} 2&1&0&0\\0&2&1&0\\0&0&2&0\\0&0&0&2 \end{pmatrix}$

3. Give a 4×4 matrix, in Jordan canonical form, that has eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 5$, such that λ_1 has both algebraic and geometric multiplicity 2, and that λ_2 has algebraic multiplicity 2 and geometric multiplicity 1.

Because λ_1, λ_2 each have algebraic multiplicity 2, the matrix must have two 2's and two 5's on the diagonal. Further, because λ_2 has geometric multiplicity 1, there is a single Jordan block for $\lambda = 5$ (thus of size 2). Because λ_1 has geometric multiplicity 2, there are two Jordan blocks for $\lambda = 2$ (thus each of size 1). Hence there are exactly three possible answers: $\begin{pmatrix} 2&0&0&0\\0&2&0&0\\0&0&5&1\\0&0&0&5 \end{pmatrix}$, $\begin{pmatrix} 2&0&0&0\\0&5&1&0\\0&0&5&0\\0&0&0&2 \end{pmatrix}$, and $\begin{pmatrix} 5&1&0&0\\0&5&0&0\\0&0&2&0\\0&0&0&2 \end{pmatrix}$.

For the last two problems, let $A = \begin{pmatrix} 1&1&1\\2&1&0\\0&1&2 \end{pmatrix}$.

4. Find the characteristic polynomial of A, and its eigenvalues.

We calculate $|tI - A| = t^3 - 4t^2 + 3t = t(t-1)(t-3)$. Hence the eigenvalues of A are $0, 1, 3$ (each with algebraic multiplicity 1).

5. For each eigenvalue of A, find a basis for the corresponding eigenspace.

For eigenvalue 0, we put $(0I - A)$ into row canonical form as $\begin{pmatrix} 1&0&-1\\0&1&2\\0&0&0 \end{pmatrix}$. This matrix has one-dimensional nullspace, with basis $\{(1, -2, 1)\}$.

For eigenvalue 1, we put $(1I - A)$ into row canonical form as $\begin{pmatrix} 1&0&0\\0&1&1\\0&0&0 \end{pmatrix}$. This matrix has one-dimensional nullspace, with basis $\{(0, 1, -1)\}$.

For eigenvalue 3, we put $(3I - A)$ into row canonical form as $\begin{pmatrix} 1&0&-1\\0&1&-1\\0&0&0 \end{pmatrix}$. This matrix has one-dimensional nullspace, with basis $\{(1, 1, 1)\}$.