Math 254 Fall 2013 Exam 11 Solutions

1. **Carefully state the definition of "span". Give two examples from within $P_1(t)$.**

   The span of a set of vectors is the set of all their linear combinations. It happens to be a vector space. Examples: $Span(\{0\}) = \{0\}, Span(\{1+t\}) = \{a + at : a \in \mathbb{R}\}, Span(\{1,t\}) = P_1(t)$.

2. **True or false: For all A, B, $det(A+B) = det(A) + det(B)$. Be sure to justify your answer.**

   False; we need one counterexample. Let $A = I_2$, $B = -I_2$. Each is triangular; we have $det(A) = 1$, while $det(B) = (-1)^2 = 1$. However $A + B = 0$ so $det(A+B) = 0$.

3. **Let $A = \left(\begin{smallmatrix} 2 & 0 \\ 3 & 1 \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} 5 & 7 \\ 0 & 1 \end{smallmatrix}\right)$. Let C be the block matrix $\left(\begin{smallmatrix} A & 0 \\ A & B \end{smallmatrix}\right)$, let F be the block matrix $\left(A^T \; B\right)$, and let D be the block matrix $\left(\begin{smallmatrix} B & F & I \\ 0 & C & F^T \\ 0 & 0 & A \end{smallmatrix}\right)$. Find $|D|$.**

   Because D is block triangular, we have $|D| = |B||C||A|$. Because A, B are triangular, we compute $|A| = 2 \cdot 1 = 2$ and $|B| = 5 \cdot 1 = 5$. Because C is block triangular, we have $|C| = |A||B| = 2 \cdot 5 = 10$. Putting it all together, $|D| = 5 \cdot 10 \cdot 2 = 100$.

4. **Determine which value(s) of a will lead to the following system having a unique solution: $\{x + 2y + az = 1, ax + ay + z = 1, x - y + az = 1\}$.**

   We need $det(A) \neq 0$, for $A = \left(\begin{smallmatrix} 1 & 2 & a \\ a & a & 1 \\ 1 & -1 & a \end{smallmatrix}\right)$. We compute $det(A) = 3 - 3a^2$ and solve $3 - 3a^2 = 0$. This has solutions $a = \pm 1$. Hence, for all *other* a, this system will have a unique solution.

5. **Use Cramer's Rule to determine which value(s) of a (if any) will lead to the system $\{x + 2y + az = 1, ax + ay + z = 1, x - y + az = 1\}$ having a unique solution in which $z = 2$.**

   We want $2 = z = \frac{det(A_z)}{det(A)} = \frac{det(A_z)}{3-3a^2}$, where we used the result from Problem 4. We have $A_z = \left(\begin{smallmatrix} 1 & 2 & 1 \\ a & a & 1 \\ 1 & -1 & 1 \end{smallmatrix}\right)$ and compute $det(A_z) = 3 - 3a$. So $2 = \frac{3(1-a)}{3(1-a^2)} = \frac{1-a}{(1-a)(1+a)} = \frac{1}{1+a}$. We cross-multiply this to find $a + 1 = \frac{1}{2}$, so there is a unique solution $a = -\frac{1}{2}$.

Extra: **Calculate $\begin{vmatrix} 2 & 2 & 1 & 3 & 4 & 1 \\ 4 & 2 & 1 & 3 & 0 & 1 \\ -1 & 2 & 0 & 1 & 1 & 2 \\ 1 & 0 & 2 & 3 & -1 & 1 \\ 1 & 2 & 1 & 1 & -1 & 2 \\ 0 & 1 & 2 & -2 & 1 & 1 \end{vmatrix}$.**

   Answer: 10