Math 254 Fall 2013 Exam 0 Solutions

1. Carefully state the definition of "polynomial space". Give two example vectors.
   The polynomial space in a variable is the set of all polynomials in that single variable. For example, if the variable is x, we have $x^2 + 2x + 3$ and $7x + 2$. (other examples are possible)

2. Carefully state the definition of "dependent". Give an example from $\mathbb{R}^3$.
   A set of vectors is dependent if there is a nondegenerate linear combination yielding the zero vector. An example is $\{u, v\}$ for $u = (1, 2, 3), v = (2, 4, 6)$, which satisfy $2u - v = 0$.
   Note: you need a *set* of vectors to build an example.

3. Consider the subset of $P_2(t)$ given by $S = \{1, t, t^2 + 1\}$. Prove that S is spanning.
   Let $p(t) = at^2 + bt + c$ be arbitrary in $P_2(t)$. Included in $Span(S)$ is the linear combination $(c - a)1 + bt + a(t^2 + 1) = p(t)$. Since $p(t)$ was arbitrary in $P_2(t)$, S is spanning.

4. Consider the subset of $\mathbb{R}^2$ given by $T = \{(x, y) : |x| = |y|\}$. Prove that T is not closed.
   Note that $u = (1, 1)$ and $v = (1, -1)$ are both in T, but $u + v = (2, 0)$ is not in T. Hence T is not closed under VA, so isn't closed. (T *is* closed under SM, but that doesn't matter)

5. Consider the linear function space in $\{x, y\}$. Prove that $\{x + y, x + 2y, x + 3y\}$ is dependent.
   A nondegenerate linear combination giving zero is $1(x + y) - 2(x + 2y) + 1(x + 3y) = 0$.
   Other, similar, combinations are possible.

Extra: Consider the subset S of $P_2(t)$, defined by $S = \{p(t) : p(1) = 0\}$.
First, prove that S is closed. Second, find a spanning set for S.
To prove closure under VA, let p, q be in S. Then $p+q$ is in $P_2(t)$, and $(p+q)(1) = p(1)+q(1) = 0 + 0 = 0$. Hence $p + q$ is in S.
To prove closure under SM, let p be in S and $a \in \mathbb{R}$. Then ap is in $P_2(t)$, and $(ap)(1) = a \cdot p(1) = a0 = 0$. Hence ap is in S.
Many spanning sets are possible; one example is $T = \{t - 1, t^2 - 1\}$. The tricky thing is to prove that T is spanning. A polynomial of degree at most 2 has at most two roots. However to be in S it must have at least one root, namely 1. Hence polynomials in S are all of the form $\alpha(t - 1)$ or $\alpha(t - 1)(t - r)$, for some real numbers α, r. The first type are all linear combinations of $t - 1$. The second type may be multiplied out as $\alpha(t^2 - (r + 1)t + r)$. We take the linear combination of T given by $1(t^2 - 1) - (r + 1)(t - 1) = t^2 - (r + 1)t + r$; multiplying by α gives the desired polynomial. Hence all of S is spanned by T.