Name:

Math 254 Fall 2012 Final Exam

Please read the following directions:
Please print your name in the space provided, using large letters, as "First LAST". Books, notes, calculators, and other aids are not permitted on this exam. Please write legibly, with plenty of white space. Please put your answers in the designated areas. Show all necessary work in your solutions; if you are unsure, show it. Cross out work you do not wish graded; incorrect work can lower your grade. All problems are worth 5-10 points; your total will be scaled to the standard 100 point scale. You have approximately 120 minutes.

1. Carefully state the definition of "dimension", as it applies to vector spaces. Give two examples, a four-dimensional one and an infinite-dimensional one.

2. Carefully state the definition of "spanning". Give two examples from $M_{2,2}(\mathbb{R})$.

3. Carefully state the definition of "linear transformation". Give two examples from $M_{2,2}(\mathbb{R})$ to $M_{2,2}(\mathbb{R})$.

4. Find all solutions to the following system:

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 4 & 1 & 5 \\ 3 & 6 & 2 & 6 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

5. Use the solution from (4) and $(1, 0, -1, 0)^T$ to find all solutions to the following system:

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 4 & 1 & 5 \\ 3 & 6 & 2 & 6 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

6. Find bases for the rowspace and columnspace of $\begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 4 & 1 & 5 \\ 3 & 6 & 2 & 6 \end{bmatrix}$.

The next three problems concern the vector space $P_2(x)$ and the linear transformation $T : P_2(x) \to P_2(x)$ given by $T(f(x)) = f(2x + 1)$.

7. Find representation $[T]_E$, for standard basis $E = \{x^2, x, 1\}$.

8. Find the rank and nullity of T.

9. Find the determinant of T.

10. Let U, V, W be vector spaces. Suppose that U is a subspace of V, and that V is a subspace of W. Prove that U is a subspace of W.

11. Let u, v be vectors in an inner product space. Let $p = proj_v(u)$, the projection of u onto v. Let $q = proj_v(p)$, the projection of p onto v. Prove that $p = q$.

12. Find a 4×4 matrix A such that $(A + 6I)^4 = 0$, but $(A + 6I)^3 \neq 0$. Be sure to justify.

The last three problems all concern the matrix $A = \left[\begin{smallmatrix} 4 & 2 \\ 2 & 1 \end{smallmatrix}\right]$.

13. Find the characteristic and minimal polynomials of A.

14. Find invertible matrices P, P^{-1} and a diagonal matrix D such that $D = P^{-1}AP$.

15. Prove or find a counterexample for the following statement: For all nonzero column vectors x, $x^T Ax > 0$.