Math 254 Fall 2012 Exam 9 Solutions

1. Carefully state the definition of "subspace". Give two examples, each within $\mathbb{R}^3$.

   A subspace of a vector space is a subset, that is itself a vector space. Many examples are possible, such as $\{(0,0,0)\}$, $\mathbb{R}^3$, $Span(\{(1,2,3)\})$, $Ker(f)$ for $f : \mathbb{R}^3 \to \mathbb{R}$ given by $f((a,b,c)) = a + b + 3c$.

2. Let A, B, C be linear transformations on finite-dimensional vector space V. Suppose that A is similar to B, and that B is similar to C. Prove that A is similar to C.

   Because A is similar to B, there is some matrix P with $A = P^{-1}BP$. Because B is similar to C, there is some matrix Q with $B = Q^{-1}CQ$. Plugging in, we get $A = P^{-1}Q^{-1}CQP = (QP)^{-1}C(QP)$. Hence there is some matrix $R = QP$ with $A = R^{-1}CR$, so A is similar to C.

For each $k \in \mathbb{R}$, we define a linear transformation $f_k : \mathbb{R}^2 \to \mathbb{R}^2$, given by $f_k((a,b)) = (2a + kb, a + 3b)$. The remaining three problems concern these functions f_k.

3. Determine the nullity of f_k, for each possible value of k.

   We determine which vectors f_k sends to $(0,0)$. Hence, $(2a+kb, a+3b) = (0,0)$ so $2a + kb = 0, a + 3b = 0$. If $k = 6$ then these are the same, so the solution space is one-dimensional and thus $nullity(f_6) = 1$. For any $k \neq 6$, the unique solution to the system is $(a,b) = (0,0)$ so $nullity(f_k) = 0$.

4. Determine the matrix representation $[f_k]_E$, for the standard basis $E = \{(1,0),(0,1)\}$.

   We seek $[f_k]_E = [[f_k(e_1)]_E[f_k(e_2)]_E] = \left[\begin{smallmatrix} 2 & k \\ 1 & 3 \end{smallmatrix}\right]$.

5. Determine the matrix representation $[f_k]_S$, for the basis $S = \{(1,2),(2,3)\}$.

   We first compute $P_{ES} = \left[\begin{smallmatrix} 1 & 2 \\ 2 & 3 \end{smallmatrix}\right]$. We now compute $P_{SE} = P_{ES}^{-1} = \left[\begin{smallmatrix} -3 & 2 \\ 2 & -1 \end{smallmatrix}\right]$. Lastly, we find $[f_k]_S = P_{SE}[f_k]_E P_{ES} = \left[\begin{smallmatrix} -3 & 2 \\ 2 & -1 \end{smallmatrix}\right] \left[\begin{smallmatrix} 2 & k \\ 1 & 3 \end{smallmatrix}\right] \left[\begin{smallmatrix} 1 & 2 \\ 2 & 3 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 8-6k & 10-9k \\ -3+4k & -3+6k \end{smallmatrix}\right]$.