Math 254 Fall 2012 Exam 6 Solutions

1. Carefully state the definition of "dimension". Give two examples that are each associated to subspaces of $\mathbb{R}^4$.

   The dimension of a vector space is the number of elements of any basis. A one-dimensional subspace of $\mathbb{R}^4$ is $Span(\{(1,2,3,4)\})$. A two-dimensional subspace of $\mathbb{R}^4$ is $Span(\{(1,2,3,4),(1,1,1,1)\})$.

2. Let V be a vector space with basis $S = \{s_1, s_2, s_3\}$. The representation (in basis S) is a map $[\ ]_S : V \to \mathbb{R}^3$. Prove that $[\ ]_S$ is a linear transformation.

   Let $u \in V$. Because S is a basis, there are some scalars a_1, a_2, a_3 where $u = a_1s_1 + a_2s_2 + a_3s_3$. Similarly, for $v \in S$ there are some scalars b_1, b_2, b_3 where $v = b_1s_1 + b_2s_2 + b_3s_3$. We have $u+v = (a_1+b_1)s_1 + (a_2+b_2)s_2 + (a_3+b_3)s_3$, hence $[u+v]_S = [u]_S + [v]_S$. Thus $[\ ]_S$ respects vector addition. For any $k \in \mathbb{R}$, $ku = (ka_1)s_1 + (ka_2)s_2 + (ka_3)s_3$, hence $[ku]_S = k[u]_S$. Thus $[\ ]_S$ respects scalar multiplication.

3. Consider the vector space $P_2(t)$. Determine if $S = \{t^2+2, t^2+t+3, t^2-t+1\}$ is independent.

   Let $E = \{t^2, t, 1\}$. We represent each element of S in basis E, then place as either rows or columns, e.g. $\begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 3 \\ 1 & -1 & 1 \end{bmatrix}$. We apply elementary row operations to place this in row canonical form, $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. This is two-dimensional, hence the rowspace of $[S]_E$ is two-dimensional, hence $Span(S)$ is two-dimensional, hence S is dependent.

4. In the vector space $\mathbb{R}^2$, set $S = \{(2,3),(5,8)\}$, a basis. Find the change-of-basis matrix from the standard basis E to S, and use this matrix to find $[(7,11)]_S$.

   We have $P_{ES} = \begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix}$, but we want $P_{SE} = P_{ES}^{-1} = \begin{bmatrix} 8 & -5 \\ -3 & 2 \end{bmatrix}$, which we can find with the row-reduction algorithm or using the 2×2 formula. $[(7,11)]_S = P_{SE}[(7,11)]_E = \begin{bmatrix} 8 & -5 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 7 \\ 11 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

5. In the vector space $M_{2,2}(\mathbb{R})$, the set of 2×2 matrices, set $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}$, $C = \begin{pmatrix} -3 & 2 \\ -2 & 3 \end{pmatrix}$. Set $V = Span(\{A, B, C\})$. Find the dimension of V, and a basis.

   We first choose a basis for $M_{2,2}$, say $E = \{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix})\}$. We represent A, B, C in this basis, then place these as either rows or columns, e.g. $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 0 & 1 & -1 \\ -3 & 2 & -2 & 3 \end{bmatrix}$. We apply elementary row operations to place this in echelon form, $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & -4 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Hence, V is two-dimensional, with basis $\{\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -4 \\ 1 & -3 \end{pmatrix}\}$.