Math 254 Fall 2012 Exam 5 Solutions

1. Carefully state the definition of "independent". Give two examples from $P_2(t)$.

   A set of vectors is independent if every nondegenerate linear combination yields a nonzero vector. Many examples are possible, such as $\{1\}, \{1, t\}, \{1+t, 1-t\}, \{1, t, t^2\}$. All correct examples are, among other things, sets of polynomials in t.

2. Let S be the set of all symmetric 2×2 matrices; it turns out that S is a subspace of $M_{2,2}(\mathbb{R})$. Find the dimension of S, and a basis for S.

   S is not all of $M_{2,2}(\mathbb{R})$, since not every matrix is symmetric, so $dim(S)$ is at most 3. If we can find a set of three elements of S that are independent, then that will prove S has dimension at least 3, hence exactly 3 (and this set will also be a basis of S). Many such sets are possible; the simplest is $\{\overline{u}, \overline{v}, \overline{w}\}$, for $\overline{u} = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right], \overline{v} = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right], \overline{w} = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. To prove this is independent, suppose linear combination $a\overline{u} + b\overline{v} + c\overline{w} = \left[\begin{smallmatrix} a & c \\ c & b \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]$; but then $a = b = c = 0$.

The last three problems all concern $A = \begin{bmatrix} 1 & -3 & 0 & 4 \\ 2 & 0 & 6 & 3 \\ 3 & 1 & 10 & 2 \\ 4 & -7 & 5 & 1 \end{bmatrix}$, which is row equivalent to $B = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

3. What can you conclude about $Span(\{(1, -3, 0, 4), (2, 0, 6, 3), (3, 1, 10, 2), (4, -7, 5, 1)\})$?

   This vector space is the rowspace of A, which coincides with the rowspace of B. Hence, a basis for this vector space is $\{(1, 0, 3, 0), (0, 1, 1, 0), (0, 0, 0, 1)\}$. In particular, this space is three-dimensional.

4. What can you conclude about $Span(\{(1, 2, 3, 4), (-3, 0, 1, -7), (0, 6, 10, 5), (4, 3, 2, 1)\})$?

   This vector space is the columnspace of A. B has pivots in the first, second, and fourth columns. Hence the first, second, and fourth elements of this set form a basis for this vector space, namely $\{(1, 2, 3, 4), (-3, 0, 1, -7), (4, 3, 2, 1)\}$. In particular, this space is three-dimensional.

5. Find a basis for the solution space of the homogeneous system of equations $A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$.

   The solution space will be one-dimensional, hence its basis will consist of a single vector. The solution space has x_3 free, and $x_1 = -3x_3, x_2 = -x_3, x_4 = 0$. Hence a basis is $\{(-3, -1, 1, 0)\}$. Other solutions are possible, but they will all be multiples of this, e.g. $\{(6, 2, -2, 0)\}$.