Math 254 Fall 2012 Exam 3 Solutions

1. Carefully state the definition of "subspace". Give two examples within $\mathbb{R}^2$.

   A subspace is a subset of a vector space, that is itself a vector space. The only zero-dimensional example is $\{(0,0)\}$; the only two-dimensional example is $\mathbb{R}^2$ itself. Many one-dimensional examples are possible, all equivalent to $\{k\bar{v} : k \in \mathbb{R}\} = Span(v)$, for some nonzero vector v.

2. Prove or provide a counterexample to the following statement:
   For all 2×2 matrices A, B, if A and B are both invertible then $A + B$ is also invertible.

   The statement is false, so we need a counterexample, i.e. specific invertible matrices A, B such that $A + B$ is not invertible. Many are possible; one is $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. A and B are each their own inverses, hence are invertible. $A+B = 0$, which is not invertible since if it were $00^{-1} = I$, but also $00^{-1} = 0$ and $0 \neq I$.

3. Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 0 & 2 & 1 \end{bmatrix}$. Find symmetric B and skew-symmetric C with $A = B + C$.

   Set $B = \frac{1}{2}(A + A^T), C = \frac{1}{2}(A - A^T)$, as given by the relevant theorem. Hence $B = \begin{bmatrix} 1 & 2 & 0.5 \\ 2 & 3 & 1.5 \\ 0.5 & 1.5 & 1 \end{bmatrix}$, and $C = \begin{bmatrix} 0 & 0 & 0.5 \\ 0 & 0 & -0.5 \\ -0.5 & 0.5 & 0 \end{bmatrix}$.

4. Find $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 0 & 2 & 1 \end{bmatrix}^{-1}$.

   $$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} \\ R_2 - 2R_1 \\ \\ \end{array} \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} R_1 + 2R_2 \\ \\ R_3 + 2R_2 \end{array} \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -3 & 2 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 0 & -1 & -4 & 2 & 1 \end{array}\right] \begin{array}{c} R_1 - R_3 \\ R_2 - R_3 \\ \\ \end{array} \longrightarrow$$
   $$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & 2 & -1 & -1 \\ 0 & 0 & -1 & -4 & 2 & 1 \end{array}\right] \begin{array}{c} \\ -R_2 \\ -R_3 \end{array} \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -2 & 1 & 1 \\ 0 & 0 & 1 & 4 & -2 & -1 \end{array}\right].$$ Hence $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 0 & 2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ -2 & 1 & 1 \\ 4 & -2 & -1 \end{bmatrix}$.

5. Write $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ as the product of elementary matrices.

   We use ERO's to turn this matrix into the identity, as $R_2 \to R_2 - 2R_1, R_1 \to R_1 + 2R_2, R_2 \to -R_2$. Using elementary matrices, this is $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Multiplying on the left by $\left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\right)^{-1}$, we get $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = \left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\right)^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, a product of elementary matrices.