Math 254 Fall 2012 Exam 11 Solutions

1. Carefully state the definition of "dependent". Give two examples from $P_2(t)$.

   A set of vectors is dependent if there is a nondegenerate linear combination of them that yields the zero vector. Many examples are possible, such as $\{0\}, \{1,3\}, \{t,2t\}, \{1,t,1+t\}, \{1,t,t^2,2+3t+4t^2\}$

The remaining problems all concern the matrix $A = \left[\begin{smallmatrix} 0 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{smallmatrix}\right]$.

2. Find the characteristic polynomial $\Delta_A(t)$ of A.

   We calculate $\Delta_A(t) = |tI - A| = \left|\left[\begin{smallmatrix} t & 1 & -1 \\ 0 & t-1 & 0 \\ -1 & 0 & t \end{smallmatrix}\right]\right| = t^3 - t^2 - t + 1$.

3. Find all the eigenvalues of A.

   We factor $\Delta_A(t) = t^3 - t^2 - t + 1$ as $(t-1)^2(t+1)$. Hence the eigenvalues of A are $\lambda = 1, -1$.

4. For each eigenvalue of A, find a maximal independent set of eigenvectors.

   For $\lambda = 1$, we compute $\lambda I - A = \left[\begin{smallmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{smallmatrix}\right]$, whose row canonical form is $\left[\begin{smallmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right]$. This has two pivots, hence a one-dimensional nullspace, with basis $\{(1,0,1)\}$. This is the desired maximal independent set of eigenvectors.
   For $\lambda = -1$, we compute $\lambda I - A = \left[\begin{smallmatrix} -1 & 1 & -1 \\ 0 & -2 & 0 \\ -1 & 0 & -1 \end{smallmatrix}\right]$, whose row canonical form is $\left[\begin{smallmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right]$. This has two pivots, hence a one-dimensional nullspace, with basis $\{(-1,0,1)\}$. This is the desired maximal independent set of eigenvectors.

5. For each eigenvalue of A, give its algebraic and geometric multiplicity. What is the Jordan form of A?

   Because $\lambda = -1$ is a single root, the algebraic and geometric multiplicities are both 1. Because $\lambda = 1$ is a double root, the algebraic multiplicity is 2 and the geometric multiplicity is either 1 or 2. By the calculation of problem 4, we see that the geometric multiplicity is actually 1. Hence A has Jordan form $\left[\begin{smallmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{smallmatrix}\right]$ or $\left[\begin{smallmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{smallmatrix}\right]$.