

## Math 254 Exam 2b Solutions

1. Carefully state the definition of “linear equation”. Give two examples, each in three variables.

A linear equation is equivalent to a linear combination of variables, set equal to a constant. Many examples are possible, such as  $x + y + z = 3$  and  $x_1 + 2x_2 - x_3 = 5$ .

The remaining problems all concern the following system.

$$\begin{aligned}x_1 + 3x_2 + 2x_3 + 4x_4 &= 10 \\2x_1 + 6x_2 + 5x_4 &= 13 \\3x_1 + 9x_2 - 10x_3 &= 2 \\x_1 + 3x_2 - 2x_3 + 6x_4 &= 8 \\2x_3 - x_4 &= 1\end{aligned}$$

2. Write the above system as a matrix equation.

This system, like all linear systems, can be written as  $Ax = b$ .

$$A = \begin{bmatrix} 1 & 3 & 2 & 4 \\ 2 & 6 & 0 & 5 \\ 3 & 9 & -10 & 0 \\ 1 & 3 & -2 & 6 \\ 0 & 0 & 2 & -1 \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, b = \begin{bmatrix} 10 \\ 13 \\ 2 \\ 8 \\ 1 \end{bmatrix}.$$

3. Write the above system as an augmented matrix. Put this in echelon form, justifying each step using elementary row operations. Using the echelon form, find the general solution to the system.

$$\begin{aligned}\begin{bmatrix} 1 & 3 & 2 & 4 & 10 \\ 2 & 6 & 0 & 5 & 13 \\ 3 & 9 & -10 & 0 & 2 \\ 1 & 3 & -2 & 6 & 8 \\ 0 & 0 & 2 & -1 & 1 \end{bmatrix} &\rightarrow \begin{bmatrix} 1 & 3 & 2 & 4 & 10 \\ 0 & 0 & -4 & -3 & -7 \\ 0 & 0 & -16 & -12 & -28 \\ 0 & 0 & -4 & 2 & -2 \\ 0 & 0 & 2 & -1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 2 & 4 & 10 \\ 0 & 0 & -4 & -3 & -7 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 5 \\ 0 & 0 & 0 & -5/2 & -5/2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 2 & 4 & 10 \\ 0 & 0 & -4 & -3 & -7 \\ 0 & 0 & 0 & 5 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \{-2R_1 + R_2 \rightarrow R_2, -3R_1 + R_3 \rightarrow R_3, -R_1 + R_4 \rightarrow R_4\}, \\ \{-4R_2 + R_3 \rightarrow R_3, -R_2 + R_4 \rightarrow R_4, -R_2/2 + R_5 \rightarrow R_5\}, \{R_4/2 + R_5 \rightarrow R_5, R_3 \leftrightarrow R_4\}\end{aligned}$$

$5x_4 = 5$ , hence  $x_4 = 1$ .  $-4x_3 - 3x_4 = -7$ , hence  $x_3 = 1$ .  $x_2$  is free.  $x_1 + 3x_2 + 2x_3 + 4x_4 = 10$ , hence  $x_1 = -3x_2 + 4$ . The general solution is  $(4 - 3x_2, x_2, 1, 1)$ .

4. Write the above system as an augmented matrix. Put this in row canonical form, justifying each step using elementary row operations. Using the row canonical form, find all solutions to the system.

We pick this up where the previous problem left off.

$$\begin{bmatrix} 1 & 3 & 2 & 4 & 10 \\ 0 & 0 & -4 & -3 & -7 \\ 0 & 0 & 0 & 5 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 2 & 4 & 10 \\ 0 & 0 & 1 & 3/4 & 7/4 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 0 & 5/2 & 13/2 \\ 0 & 0 & 1 & 3/4 & 7/4 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\{-R_3/4 \rightarrow R_3, R_5/5 \rightarrow R_5\}, \{-2R_2 + R_1 \rightarrow R_1\}, \{-(5/2)R_3 + R_1 \rightarrow R_1, -(3/4)R_3 + R_2 \rightarrow R_2\}$

It is now easy to read off  $x_1 = 4 - 3x_2, x_3 = 1, x_4 = 1$ , yielding the same general solution.

5. Write the homogeneous system associated to the above system. Solve this homogeneous system. Then use the particular solution  $(1, 1, 1, 1)$  to give the general solution to the original system.

The homogeneous system is  $Ax = 0 = (0, 0, 0, 0, 0)^T$ . Its augmented matrix is identical to the one above, except that the rightmost column is all zeroes.

$$\begin{bmatrix} 1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The general solution to the homogeneous system is  $x_4 = x_3 = 0, x_1 = -3x_2$ , i.e.  $(-3a, a, 0, 0)$ . We add this to the specific solution  $(1, 1, 1, 1)$  to get the general solution  $(1 - 3a, 1 + a, 1, 1)$ .