Math 254 Exam 11 Solutions

1. Carefully state the definition of "linear mapping". Give two examples, each from $\mathbb{R}^2$ to $P_2(t)$.

   A linear mapping is a function $f : V \to W$ between two vector spaces, that satisfies $f(u+v) = f(u) + f(v)$ and $f(ku) = kf(u)$ for all vectors u, v and all scalars k. Many examples are possible, such as $f(a, b) = at^2 + bt$, $g(a, b) = 0$, $h(a, b) = bt + 3b$.

The remaining problems all concern the matrix $A = \begin{bmatrix} 3 & 1 & -5 \\ -1 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix}$.

2. Find the characteristic polynomial $\Delta(t)$ of A.

   We calculate $|tI - A| = \begin{vmatrix} t-3 & -1 & 5 \\ 1 & t-1 & -1 \\ 0 & 0 & t+2 \end{vmatrix}$. Expanding on the last row, $|tI - A| = (t+2)(-1)^6 \begin{vmatrix} t-3 & -1 \\ 1 & t-1 \end{vmatrix} = (t + 2)[(t - 3)(t - 1) + 1] = (t + 2)[t^2 - 4t + 3 + 1] = (t + 2)(t - 2)^2 = t^3 - 2t^2 - 4t + 8$.

3. Find all the eigenvalues of A. What are their algebraic multiplicities?

   If we calculated $\Delta(t)$ as $(t + 2)(t - 2)^2$, it is easy to see that $\lambda = -2, \lambda = 2$ are the two eigenvalues, with algebraic multiplicities 1,2 respectively. If we calculated $\Delta(t)$ using a formula, or multiplied it out, we would first need to factor it.

4. For each eigenvalue of A, find a maximal independent set of eigenvectors.

   For $\lambda = -2$, we seek the nullspace of $A - \lambda I = \begin{bmatrix} 5 & 1 & -5 \\ -1 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. We put in row echelon form as $R1 = R1 + 5R2, R2 \leftrightarrow R1$, getting $\begin{bmatrix} -1 & 3 & 1 \\ 0 & 16 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Since this has two pivots, the nullspace (i.e. the eigenspace of $\lambda = -2$) has dimension 1. We may choose any nonzero vector from the nullspace, such as $(1, 0, 1)$.

   For $\lambda = 2$, we seek the nullspace of $A - \lambda I = \begin{bmatrix} 1 & 1 & -5 \\ -1 & -1 & 1 \\ 0 & 0 & -4 \end{bmatrix}$. We put in row echelon form as $R1 = R1 + R2, R3 = R3 - R2$, getting $\begin{bmatrix} -1 & 1 & -5 \\ 0 & 0 & -4 \\ 0 & 0 & 0 \end{bmatrix}$. Since this has two pivots, the nullspace (i.e. the eigenspace of $\lambda = 2$) has dimension 1. We may choose any nonzero vector from the nullspace, such as $(1, -1, 0)$.

5. Find the minimal polynomial $m(t)$ of A. Is A diagonalizable?

   We must have either $m(t) = (t + 2)(t - 2)$ or $m(t) = (t + 2)(t - 2)^2 = \Delta(t)$.

   Method 1: If $m(t) = (t + 2)(t - 2) = t^2 - 4$, then $A^2 - 4I = 0$. We calculate $A^2 - 4I = \begin{bmatrix} 3 & 1 & -5 \\ -1 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 3 & 1 & -5 \\ -1 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 4 & -4 \\ -4 & -4 & 4 \\ 0 & 0 & 0 \end{bmatrix}$. Since this is not the zero matrix, $m(t)$ is not $(t + 2)(t - 2)$, and hence $m(t) = \Delta(t)$. Also, since $m(t)$ is not the product of distinct linear terms, A is not diagonalizable.

   Method 2: A is not diagonalizable, since the algebraic multiplicity of $\lambda = 2$ is 2 while the geometric multiplicity is 1. Hence $m(t)$ cannot be $(t + 2)(t - 2)$, and hence $m(t) = \Delta(t)$.