Math 254 Fall 2011 Exam 0 Solutions

1. Carefully state the definition of "dependent". Give two examples in $\mathbb{R}^3$.

   A set of vectors is dependent if there is a nondegenerate linear combination yielding the zero vector. Many examples are possible, such as $\{(0,0,0)\}$ or $\{(1,2,0),(2,4,0)\}$.

2. Carefully state the definition of "spanning". Give two examples in $\mathbb{R}^2$.

   A set of vectors is spanning if every vector in the vector space may be achieved as a linear combination of the vectors in the set. Many examples are possible, such as $\{(1,0),(0,1)\}$ or $\{(1,0),(0,1),(1,1)\}$.

3. Consider the set S of all $\bar{v}=(v_1,v_2)$ such that $v_1 \geq v_2$. This is a subset of $\mathbb{R}^2$. Prove that this is not a subspace of $\mathbb{R}^2$.

   This requires finding one counterexample. It is not possible to find one for vector addition, because in fact S is closed under vector addition. Here is one possible example for scalar multiplication: $a=-1, \bar{u}=(1,0)$. $\bar{u}$ is in S, but $a\bar{u}=(-1,0)$ is not.

4. Consider $f:\mathbb{R}^3 \to \mathbb{R}$ given by $f((x_1,x_2,x_3))=2x_2-x_3$. Prove that this is a linear transformation.

   We must prove that f respects scalar multiplication and vector addition. First, let $\bar{x}=(x_1,x_2,x_3)$ be an arbitrary vector from $\mathbb{R}^3$, and let $a \in \mathbb{R}$. $f(a\bar{x})=f((ax_1,ax_2,ax_3))=2ax_2-ax_3=a(2x_2-x_3)=af(\bar{x})$. Second, let $\bar{y}=(y_1,y_2,y_3)$ be another arbitrary vector from $\mathbb{R}^3$. $f(\bar{x}+\bar{y})=f((x_1+y_1,x_2+y_2,x_3+y_3))=2(x_2+y_2)-(x_3+y_3)=(2x_2-x_3)+(2y_2-y_3)=f(\bar{x})+f(\bar{y})$.

5. Prove that $\{(1,1),(0,1)\}$ is a basis of $\mathbb{R}^2$.

   We must prove that this set is independent and spanning. First, suppose $a(1,1)+b(0,1)=(0,0)$. This means that $a+0b=0, a+b=0$. The only solution is $a=b=0$, so this set is independent. Second, let (x,y) be arbitrary in $\mathbb{R}^2$. We have the linear combination $x(1,1)+(y-x)(0,1)=(x,x)+(0,y-x)=(x,y)$. Hence this set is spanning.

   ALTERNATE SOLUTION: We know $\mathbb{R}^2$ has two dimensions, so we may use this fact together with either of the above. If we prove the set is independent, then it must be spanning since its span has dimension 2 and it lives inside $\mathbb{R}^2$. If we prove the set is spanning, then it must be independent since no smaller set could be spanning inside $\mathbb{R}^2$.