Math 254-1 Exam 9 Solutions

1. Carefully define the term "spanning". Give two examples in $\mathbb{R}^2$.

   A set of vectors is spanning if every vector in the vector space can be expressed as a linear combination of vectors from this set. Many examples are possible, e.g. $\{(1,0),(0,1)\}, \{(1,0),(0,1),(1,1),(2,3)\}$.

2. Consider the basis $S = \{(1,-2),(2,-5)\}$ of $\mathbb{R}^2$, and the linear operator $F(x,y) = (2x+3y, 4x-5y)$. Find the matrix representation $[F]_S$.

   We have $P_{ES} = \begin{pmatrix} 1 & 2 \\ -2 & -5 \end{pmatrix}$, so $P_{SE} = P_{ES}^{-1} = \begin{pmatrix} 5 & 2 \\ -2 & -1 \end{pmatrix}$. We calculate $[F]_E = ([F(e_1)]_E \ [F(e_2)]_E) = \left([\begin{pmatrix} 2 \\ 4 \end{pmatrix}]_E \ [\begin{pmatrix} 3 \\ -5 \end{pmatrix}]_E\right) = \begin{pmatrix} 2 & 3 \\ 4 & -5 \end{pmatrix}$. Hence $[F]_S = P_{SE}[F]_E P_{ES} = \begin{pmatrix} 5 & 2 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 4 & -5 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -2 & -5 \end{pmatrix} = \begin{pmatrix} 8 & 11 \\ -6 & -11 \end{pmatrix}$

3. Prove that, for any square matrices A, B, if A is similar to B, then B must be similar to A.

   Suppose that A is similar to B. Then there is some invertible matrix P with $A = PBP^{-1}$. Multiply this expression on the left by P^{-1}, and on the right by P, to get $P^{-1}AP = P^{-1}PBP^{-1}P = IBI = B$. Hence, there is some invertible matrix $Q = P^{-1}$, such that $B = QAQ^{-1}$, so B is similar to A.

For the last two questions, set V to be the vector space of functions that have as a basis $S = \{1, \sin\theta, \cos\theta, \sin 3\theta, \cos 3\theta\}$.

4. Let D be the differential operator on V, $D(f(\theta)) = f'(\theta)$. Find the matrix representation $[D]_S$.

   $$[D]_S = ([D(1)]_S \ [D(\sin\theta)]_S \ [D(\cos\theta)]_S \ [D(\sin 3\theta)]_S \ [D(\cos 3\theta)]_S) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 3 & 0 \end{pmatrix}.$$

5. Let L be the operator on V given by $L(f(\theta)) = f''(\theta) - f(\theta)$. Find the matrix representation $[L]_S$.

   $$[L]_S = ([L(1)]_S \ [L(\sin\theta)]_S \ [L(\cos\theta)]_S \ [L(\sin 3\theta)]_S \ [L(\cos 3\theta)]_S) = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & -10 & 0 \\ 0 & 0 & 0 & 0 & -10 \end{pmatrix}.$$