Math 254-2 Exam 7 Solutions

1. Carefully define the linear algebra term "degenerate". Give two examples.

   A linear combination is degenerate if each coefficient is zero. Many examples are possible, such as $0x + 0y$ or $0(1,2) + 0(-1,3)$.

2. Carefully define the linear algebra term "inner product". Give two examples on $\mathbb{R}^2$.

   An inner product is a function that, given two vectors, yields one real number. It must satisfy three properties; each must hold for all vectors u, v, w and all scalars a, b: (I1) $\langle au + bw, v\rangle = a\langle u, v\rangle + b\langle w, v\rangle$, (I2) $\langle u, v\rangle = \langle v, u\rangle$, and (I3) $\langle u, u\rangle \geq 0$, with equality precisely when $u = 0$. Many examples on $\mathbb{R}^2$ are possible, since $\langle u, v\rangle = u^T Av$ is an inner product for every positive definite matrix A. For $A = I$, this is the usual dot product. Other possible A include $\left(\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2 & 3 \\ 1 & 2 \end{smallmatrix}\right)$. A 2×2 matrix is positive definite precisely when the determinant is positive AND the diagonal entries are positive.

The remaining three problems all concern the vector space $M_{2,2}(\mathbb{R})$ with standard inner product $<A, B>= tr(B^T A)$.

3. Find all values of k such that $\left(\begin{smallmatrix} 1 & 2 \\ 2 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ k & 3 \end{smallmatrix}\right)$ are orthogonal.

   These vectors are orthogonal precisely when $\langle \left(\begin{smallmatrix} 1 & 2 \\ 2 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ k & 3 \end{smallmatrix}\right)\rangle = 0$. $\langle \left(\begin{smallmatrix} 1 & 2 \\ 2 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ k & 3 \end{smallmatrix}\right)\rangle = tr\left(\left(\begin{smallmatrix} 1 & k \\ 0 & 3 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 2 \\ 2 & 1 \end{smallmatrix}\right)\right) = tr\left(\left(\begin{smallmatrix} 2k+1 & k+2 \\ 6 & 3 \end{smallmatrix}\right)\right) = 4 + 2k$. This has exactly one solution, namely $k = -2$.

4. Find a basis for $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 3 \end{smallmatrix}\right)^\perp$.

   For convenience, set $S = Span\left\{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 3 \end{smallmatrix}\right)^\perp\right\}$. Because $M_{2,2}(\mathbb{R}) = Span\left\{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 3 \end{smallmatrix}\right)\right\} \oplus S$, we have $dim\ M_{2,2}(\mathbb{R}) = dim\ Span\left\{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 3 \end{smallmatrix}\right)\right\} + dim\ S$, and hence $dim\ S = 3$. So we need three linearly independent vectors, each orthogonal to $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 3 \end{smallmatrix}\right)$. Among the standard vectors, $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ both work. One more is, for example, $\left(\begin{smallmatrix} 3 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. The set $\left\{\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 3 & 0 \\ 0 & -1 \end{smallmatrix}\right)\right\}$ is linearly independent, since each vector has a coordinate that the other two do not, hence only the degenerate linear combination yields $\bar{0}$. Alternate proof of independence strategy: Represent these vectors using the standard basis (as columns), put them together into a 4×3 matrix, put this in row echelon form, and observe there are three pivots.

5. Use the Gram-Schmidt process to find an orthogonal basis for the space $Span\left\{\left(\begin{smallmatrix} 1 & 2 \\ 2 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & 3 \end{smallmatrix}\right)\right\}$.

   Set $v_1 = \left(\begin{smallmatrix} 1 & 2 \\ 2 & 1 \end{smallmatrix}\right), v_2 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 3 \end{smallmatrix}\right)$. We take $w_1 = v_1$, then $w_2 = v_2 - \frac{\langle v_2, w_1\rangle}{\langle w_1, w_1\rangle} w_1$. We calculate $\langle v_2, w_1\rangle = \langle v_2, v_1\rangle = 4, \langle w_1, w_1\rangle = 10$, so $u_2 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 3 \end{smallmatrix}\right) - 0.4\left(\begin{smallmatrix} 1 & 2 \\ 2 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0.6 & -0.8 \\ -0.8 & 2.6 \end{smallmatrix}\right)$.