Math 254-2 Exam 4 Solutions

1. Carefully state the definition of "subspace". Give two examples from $\mathbb{R}^2$.

   A subspace is a vector space, that is contained within another vector space. Many examples are possible from $\mathbb{R}^2$: $\{\bar{0}\}$, $\mathbb{R}^2$ itself, $Span(\bar{v})$ for any vector $\bar{v}$ (in $\mathbb{R}^2$), the solution set to any 2×2 homogeneous system of linear equations. Note that $\mathbb{R}^1$ is *NOT* a subspace, since none of its vectors are in $\mathbb{R}^2$.

2. Carefully state five of the eight vector space axioms.

   It is important not only to have the axioms right, but the quantifiers (for all vectors $\bar{u}, \bar{v}$, etc.) You may find a list of the axioms on p.152 of the text. The names the book gives them (e.g. A_3) are unimportant.

3. Let $S = \{f(x) : f(17) = 0\} \subseteq \mathbb{R}[x]$ be the set of all polynomials that are zero at $x = 17$. Prove that this is a vector space.

   S is a nonempty subset of $\mathbb{R}[x]$, so by Thm. 4.2 we need only check closure. If f, g are both in S, then $f(17) = 0 = g(17)$. $(f+g)(17) = f(17) + g(17) = 0 + 0 = 0$, so $f + g$ is in S. If f is in S, and a is any scalar, then $(cf)(17) = cf(17) = c0 = 0$, so cf is in S. S satisfies closure, hence is a subspace.

   Alternate proof: Instead of two steps, closure can be verified in one step. If f, g are both in S, and k, k' are any scalars, then $(kf + k'g)(17) = kf(17) + k'g(17) = k0 + k'0 = 0$.

4. Determine, with justification, whether $(1,1,1)$ is in $Span(S)$, for $S = \{(1,2,1), (0,3,2), (2,1,0)\}$.

   The answer is yes, precisely when there are solutions to the linear system $\begin{bmatrix} 1 & 0 & 2 \\ 2 & 3 & 1 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. We solve this in the usual way, with the augmented matrix
   $\begin{bmatrix} 1 & 0 & 2 & 1 \\ 2 & 3 & 1 & 1 \\ 1 & 2 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 3 & -3 & -1 \\ 0 & 2 & -2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & -1/3 \\ 0 & 2 & -2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & -1/3 \\ 0 & 0 & 0 & 2/3 \end{bmatrix}$
   The last equation is $0 = 2/3$, which has no solutions. Hence $(1,1,1)$ is *NOT* in $Span(S)$. There is no linear combination of the elements of S, that yields $(1,1,1)$.

5. Let $W_1 = Span(S)$, for $S = \{\left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix}\right)\}$. Let $W_2 = Span(T)$, for $T = \{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)\}$. Prove that $W_1 \oplus W_2 = M_{22}(\mathbb{R})$ (the set of all 2×2 matrices).

   Note that $W_1 = \{w\left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right) + x\left(\begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix}\right)\} = \{\left(\begin{smallmatrix} w & w \\ x & x \end{smallmatrix}\right)\}, W_2 = \{y\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) + z\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)\} = \{\left(\begin{smallmatrix} y & 0 \\ z & 0 \end{smallmatrix}\right)\}$.

   Solution 1: We need to express every $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ uniquely as a sum of some vector from W_1 and some vector from W_2. We have $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} w & w \\ x & x \end{smallmatrix}\right) + \left(\begin{smallmatrix} y & 0 \\ z & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} w+y & w \\ x+z & x \end{smallmatrix}\right)$. The equations we get are $w + y = a, w = b, x + z = c, x = d$. These equations have a *unique* solution, namely $w = b, y = a - b, x = d, z = c - d$; hence $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} b & b \\ d & d \end{smallmatrix}\right) + \left(\begin{smallmatrix} a-b & 0 \\ c-d & 0 \end{smallmatrix}\right)$, where the first matrix is in W_1 and the second is in W_2.

   Solution 2: We use Thm 4.11, which requires two things: (1) $M_{22} = W_1 + W_2$, and (2) $W_1 \cap W_2 = \{\bar{0}\}$. To prove (1), we use the calculation from the first solution, although it is no longer important to have a unique decomposition. To prove (2), we need to find all matrices common to both W_1, W_2. $\left(\begin{smallmatrix} w & w \\ x & x \end{smallmatrix}\right) = \left(\begin{smallmatrix} y & 0 \\ z & 0 \end{smallmatrix}\right)$. The only solution is $w = x = y = z = 0$, hence $W_1 \cap W_2 = \{\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)\} = \{\bar{0}\}$.