# Math 254-1 Exam 4 Solutions

1. Carefully state the definition of "subspace". Give two examples from $\mathbb{R}^2$.

   A subspace is a vector space, that is contained within another vector space. Many examples are possible from $\mathbb{R}^2$: $\{\bar{0}\}$, $\mathbb{R}^2$ itself, $Span(\bar{v})$ for any vector $\bar{v}$ (in $\mathbb{R}^2$), the solution set to any $2\times 2$ homogeneous system of linear equations. Note that $\mathbb{R}^1$ is *NOT* a subspace, since none of its vectors are in $\mathbb{R}^2$.

2. Carefully state five of the eight vector space axioms.

   It is important not only to have the axioms right, but the quantifiers (for all vectors $\bar{u},\bar{v}$, etc.) You may find a list of the axioms on p.152 of the text. The names the book gives them (e.g. $A_3$) are unimportant.

3. Let $S=\{f(x): f(3)=0\}\subseteq \mathbb{R}[x]$ be the set of all polynomials that are zero at $x=3$. Prove that this is a vector space.

   $S$ is a nonempty subset of $\mathbb{R}[x]$, so by Thm. 4.2 we need only check closure. If $f,g$ are both in $S$, then $f(3)=0=g(3)$. $(f+g)(3)=f(3)+g(3)=0+0=0$, so $f+g$ is in $S$. If $f$ is in $S$, and $a$ is any scalar, then $(cf)(3)=cf(3)=c0=0$, so $cf$ is in $S$. $S$ satisfies closure, hence is a subspace.

   Alternate proof: Instead of two steps, closure can be verified in one step. If $f,g$ are both in $S$, and $k,k'$ are any scalars, then $(kf+k'g)(3)=kf(3)+k'g(3)=k0+k'0=0$.

4. Determine, with justification, whether $(1,1,1)$ is in $Span(S)$, for $S=\{(1,2,3),(2,0,1),(-3,2,1)\}$.

   The answer is yes, precisely when there are solutions to the linear system $\begin{bmatrix}1&2&-3\\2&0&2\\3&1&1\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}=\begin{bmatrix}1\\1\\1\end{bmatrix}$. We solve this in the usual way, with the augmented matrix $\begin{bmatrix}1&2&-3&1\\2&0&2&1\\3&1&1&1\end{bmatrix}\to\begin{bmatrix}1&2&-3&1\\0&-4&8&-1\\0&-5&10&-2\end{bmatrix}\to\begin{bmatrix}1&2&-3&1\\0&1&-2&1/4\\0&-5&10&-2\end{bmatrix}\to\begin{bmatrix}1&2&-3&1\\0&1&-2&1/4\\0&0&0&-3/4\end{bmatrix}$

   The last equation is $0=-3/4$, which has no solutions. Hence $(1,1,1)$ is *NOT* in $Span(S)$. There is no linear combination of the elements of $S$, that yields $(1,1,1)$.

5. Let $W_1=Span(S)$, for $S=\{\left(\begin{smallmatrix}1&1\\0&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&0\\1&1\end{smallmatrix}\right)\}$. Let $W_2=Span(T)$, for $T=\{\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)\}$. Prove that $W_1\oplus W_2=M_{22}(\mathbb{R})$ (the set of all $2\times 2$ matrices).

   Note that $W_1=\{w\left(\begin{smallmatrix}1&1\\0&0\end{smallmatrix}\right)+x\left(\begin{smallmatrix}0&0\\1&1\end{smallmatrix}\right)\}=\{\left(\begin{smallmatrix}w&w\\x&x\end{smallmatrix}\right)\}, W_2=\{y\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)+z\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)\}=\left\{\left(\begin{smallmatrix}y&0\\z&0\end{smallmatrix}\right)\right\}$.

   Solution 1: We need to express every $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$ uniquely as a sum of some vector from $W_1$ and some vector from $W_2$. We have $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)=\left(\begin{smallmatrix}w&w\\x&x\end{smallmatrix}\right)+\left(\begin{smallmatrix}y&0\\z&0\end{smallmatrix}\right)=\left(\begin{smallmatrix}w+y&w\\x+z&x\end{smallmatrix}\right)$. The equations we get are $w+y=a, w=b, x+z=c, x=d$. These equations have a *unique* solution, namely $w=b, y=a-b, x=d, z=c-d$; hence $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)=\left(\begin{smallmatrix}b&b\\d&d\end{smallmatrix}\right)+\left(\begin{smallmatrix}a-b&0\\c-d&0\end{smallmatrix}\right)$, where the first matrix is in $W_1$ and the second is in $W_2$.

   Solution 2: We use Thm 4.11, which requires two things: (1) $M_{22}=W_1+W_2$, and (2) $W_1\cap W_2=\{\bar{0}\}$. To prove (1), we use the calculation from the first solution, although it is no longer important to have a unique decomposition. To prove (2), we need to find all matrices common to both $W_1,W_2$. $\left(\begin{smallmatrix}w&w\\x&x\end{smallmatrix}\right)=\left(\begin{smallmatrix}y&0\\z&0\end{smallmatrix}\right)$. The only solution is $w=x=y=z=0$, hence $W_1\cap W_2=\{\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\}=\{\bar{0}\}$.