Math 254-2 Exam 2a Solutions

1. Carefully state the definition of "linear equation". Give two examples, one in standard form and one NOT in standard form.

   A linear equation is a linear combination of variables, set equal to a constant. In standard form the linear combination and the constant are on opposite sides of the equal sign. $3x+2y=1$ is in standard form, $3x+2y-2=0$ is not in standard form.

2. Solve the following system, using back-substitution.
$$\begin{aligned} 3x_1+4x_2+5x_3+\ x_4 &= 6\\ 7x_2-8x_3-\ x_4 &= 3\\ 9x_3+2x_4 &= 1\\ 20x_4 &= 100\end{aligned}$$

   First, $20x_4=100$ implies that $x_4=5$. Then, $9x_3+2(5)=1$ implies that $x_3=-1$. Then, $7x_2-8(-1)-(5)=3$ implies that $x_2=0$. Finally, $3x_1+4(0)+5(-1)+5=6$ implies that $x_1=2$. Hence, there is exactly one solution: $(2,0,-1,5)$.

3. Give three examples of 2×2 systems of linear equations. One should have no solutions, one should have one solution, and one should have infinitely many solutions. Demonstrate each system geometrically, and find all solutions algebraically.

   Many correct answers are possible. To have no solutions, the graph should have two parallel lines; for example $y-x=0, y-x=1$. To have one solution, the graph should have two intersecting lines; for example $y-x=0, y+x=0$, which has unique solution $x=y=0$. Two have infinitely many solutions, the graph should have two coinciding lines; for example $y-x=0, 2y-2x=0$, which has infinitely many solutions (a,a) for every real a.

4. Find the line of best fit for the following set of points: $\{(0,0),(3,0),(0,4),(2,9)\}$.

   We first calculate $N=4, \sum x=5, \sum y=13, \sum xy=18, \sum x^2=13$. This gives us the system $4b+5m=13, 5b+13m=18$. Calculating $4L_2-5L_1$, we get $27m=7$, so $m={}^7/_{27}$ and $4b+\frac{35}{27}=13$, so $b={}^{79}/_{27}$. Hence the best-fit line is $y={}^7/_{27}x+{}^{79}/_{27}$, or $27y=7x+79$.

5. Solve the following system of linear equations using Gaussian elimination and back-substitution.
$$\begin{aligned} 3x+\ y+2z &= 1\\ -6x-3z &= 1\\ 9x-4y-10z &= 5\end{aligned}$$

   We begin by $2L_1+L_2\to L_2, -3L_1+L_3\to L_3$ to get $\begin{aligned} 3x+\ y+2z &= 1\\ 2y+\ z &= 3\\ -7y-16z &= 2\end{aligned}$. Next we have $7L_2+2L_3\to L_3$ to get $\begin{aligned} 3x+\ y+2z &= 1\\ 2y+\ z &= 3\\ -25z &= 25\end{aligned}$. This is ready for back-substitution; $z=-1$, then $2y-1=3$ so $y=2$, then $3x+2+2(-1)=1$ so $x={}^1/_3$. Hence the solution is $({}^1/_3,2,-1)$.