Math 254-1 Exam 10 Solutions

1. Carefully define the term "dependent". Give two examples in $\mathbb{R}^2$.

   A set of vectors is dependent if there is a nondegenerate linear combination yielding the zero vector. Examples in $\mathbb{R}^2$ include $\{(0,0)\}$, $\{(1,1),(2,2)\}$, $\{(1,0),(0,1),(2,3)\}$.

For the next two problems, consider the matrix $A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 0 & 2 \\ 0 & 2 & -3 \end{pmatrix}$.

2. Calculate $|A|$ by using the formula for 3×3 determinants.

   We have $|A| = (2)(0)(-3) + (1)(2)(0) + (-1)(1)(2) - (0)(0)(-1) - (2)(2)(2) - (-3)(1)(1) = -2 - 8 + 3 = -7$

3. Calculate $|A|$ by expanding on the second column.

   We have $|A| = 1(-1)^{1+2}\begin{vmatrix} 1 & 2 \\ 0 & -3 \end{vmatrix} + 0(-1)^{2+2}\begin{vmatrix} 2 & -1 \\ 0 & -3 \end{vmatrix} + 2(-1)^{3+2}\begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & 2 \\ 0 & -3 \end{vmatrix} - 2\begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} = -(-3-0) - 2(4-(-1)) = -(-3) - 2(5) = -7$.

4. Solve the linear system $\left\{ \begin{matrix} 2x+y & = 5 \\ -2x+y & = 1 \end{matrix} \right\}$ using Cramer's rule.

   Cramer's rule gives $x = \frac{\begin{vmatrix} 5 & 1 \\ 1 & 1 \end{vmatrix}}{\begin{vmatrix} 2 & 1 \\ -2 & 1 \end{vmatrix}} = \frac{4}{4} = 1, y = \frac{\begin{vmatrix} 2 & 5 \\ -2 & 1 \end{vmatrix}}{\begin{vmatrix} 2 & 1 \\ -2 & 1 \end{vmatrix}} = \frac{12}{4} = 3$.

5. Find $|B|$, for $B = \begin{pmatrix} 2 & -1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 & -1 \\ 0 & 1 & 0 & -2 & 0 \\ 4 & 0 & 1 & 0 & 1 \\ 1 & 1 & -2 & 0 & 3 \end{pmatrix}$.

   Many approaches are possible. To do this efficiently requires a combination of elementary row/column operations and Laplace expansions. Adding twice the second column to the fourth gives the matrix $C = \begin{pmatrix} 2 & -1 & 0 & -1 & 0 \\ 0 & 0 & 2 & 1 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 4 & 0 & 1 & 0 & 1 \\ 1 & 1 & -2 & 2 & 3 \end{pmatrix}$. $|B| = |C|$, and we find $|C|$ by expanding on the third row: $|C| = 1C_{32}$, where $C_{32} = (-1)^{3+2}|D|$, for $D = \begin{pmatrix} 2 & 0 & -1 & 0 \\ 0 & 2 & 1 & -1 \\ 4 & 1 & 0 & 1 \\ 1 & -2 & 2 & 3 \end{pmatrix}$. We add twice the third column of D to the first column to get the matrix $E = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 2 & 2 & 1 & -1 \\ 4 & 1 & 0 & 1 \\ 5 & -2 & 2 & 3 \end{pmatrix}$. $|D| = |E|$, and we find $|E|$ by expanding on the first row: $|E| = (-1)E_{13}$, where $E_{13} = (-1)^{1+3}|F|$, for $F = \begin{pmatrix} 2 & 2 & -1 \\ 4 & 1 & 1 \\ 5 & -2 & 3 \end{pmatrix}$. We now calculate $|F| = 6 + 10 + 8 - (-5) - (-4) - 24 = 9$. Hence $|E| = -9, |C| = 9, |B| = 9$.