Math 254 Exam 5 Solutions

1. Carefully define the Linear Algebra term "independent".

   A set of vectors is independent if there is no nondegenerate linear combination of them that yields the zero vector.

   The next three questions concern the matrix $A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 2 & 6 & -3 & -3 \\ 3 & 10 & -6 & -5 \end{bmatrix}$.

   All three questions benefit from reducing A to row echelon form. We do $-2R_1 + R2 \rightarrow R_2, -3R_1 + R_3 \rightarrow R_3, -2R_2 + R_3 \rightarrow R_3$. This is actually Example 5.5 in the text, on p.189 (except for a typo; the last row of the intermediate step should be 0 4 -6 -2).

   The row echelon form is $B = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 2 & -3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

2. Set $S = rowspace(A)$. Find a basis for S. Determine the dimension of S.

   Because B has two nonzero rows, the rowspace has dimension 2. A basis is $\{(1, 2, 0, -1), (0, 2, -3, -1)\}$.

3. Set $T = columnspace(A)$. Find a basis for T. Determine the dimension of T.

   Because B has two pivots, the columnspace has dimension 2 (also because of the answer to the previous question). Because the pivots are in the first two columns, a basis for the columnspace is $\{(1, 2, 3), (2, 6, 10)\}$.

4. Write the matrix equation $AX = 0$ as a homogeneous system of equations; find a basis for its solution space.

   $$\begin{array}{rrrrcl} x + & 2y + & 0z - & w & = & 0 \\ 2x + & 6y - & 3z - & 3w & = & 0 \\ 3x + & 10y - & 6z - & 5w & = & 0 \end{array}$$

   We look at B; z, w are free variables, hence the solution space is 2 dimensional. To find a basis, try $z = 1, w = 0 : (-3, 3/2, 1, 0)$ and $z = 0, w = 1 : (0, 1/2, 0, 1)$; hence $\{(-3, 3/2, 1, 0), (0, 1/2, 0, 1)\}$ is a basis for the solution space.

5. In the vector space $\mathbb{R}^3$, set $U = span\{(1, 2, 3), (5, -2, 3)\}, V = span\{(-1, 2, 1)\}$. Determine the dimensions of each of $U, V, U \cap V, U + V$. Justify your answers.

   dim $U = 2$ since the two basis vectors are not scalar multiples of each other. dim $V = 1$.

   To find dim $(U+V)$, we put $\begin{bmatrix} 1 & 2 & 3 \\ 5 & -2 & 3 \\ -1 & 2 & 1 \end{bmatrix}$ into row echelon form: $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -12 & -12 \\ 0 & 0 & 0 \end{bmatrix}$.

   Hence dim $(U + V) = 2$, and therefore the line V actually lies in the plane U. This means that $V = U \cap V$, so dim $(U \cap V) = 1$. Alternatively, by Theorem 5.9, dim $(U + V) =$ dim $U+$ dim $V-$ dim $(U \cap V)$. We have $2 = 2 + 1-$ dim $(U \cap V)$; we solve this to get dim $(U \cap V) = 1$.