Math 254 Exam 2b Solutions

1. Carefully define the term "dimension" as it applies to vector spaces. Give two examples: a three-dimensional vector space, and an infinite-dimensional vector space.

   The dimension of a vector space is the number of elements in a basis. $\mathbb{R}^3$, the set of all 3-vectors, is a three-dimensional vector space. $\mathbb{R}[x]$, the set of all polynomials with real coefficients, is an infinite-dimensional vector space.

2. Write the system as a matrix equation.

$$\begin{bmatrix} 1 & 3 & -3 & 0 \\ 0 & 0 & 1 & 3 \\ 3 & 9 & -2 & 5 \\ 2 & 6 & 3 & -1 \\ 5 & 15 & 0 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -4 \\ -1 \\ -3 \\ 11 \\ 17 \end{bmatrix}$$

3. Write the system as an augmented matrix; put this matrix in echelon form. Justify each step using elementary row operations. Using the echelon form, find all solutions to the system (if any).

$$\left[\begin{array}{cccc|c} 1 & 3 & -3 & 0 & -4 \\ 0 & 0 & 1 & 3 & -1 \\ 3 & 9 & -2 & 5 & -3 \\ 2 & 6 & 3 & -1 & 11 \\ 5 & 15 & 0 & -7 & 17 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 3 & -3 & 0 & -4 \\ 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & 7 & 5 & 9 \\ 0 & 0 & 9 & -1 & 19 \\ 0 & 0 & 15 & -7 & 37 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 3 & -3 & 0 & -4 \\ 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & -16 & 16 \\ 0 & 0 & 0 & -28 & 28 \\ 0 & 0 & 0 & -52 & 52 \end{array}\right] \to$$

$$\to \left[\begin{array}{cccc|c} 1 & 3 & -3 & 0 & -4 \\ 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & -16 & 16 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

   The first step combines $-3R_1 + R_3 \to R_3, -2R_1 + R_4 \to R_4, -5R_1 + R_5$. The second step combines $-7R_2 + R_3 \to R_3, -9R_2 + R_4 \to R_4, -15R_2 + R_5 \to R_5$. The third step combines $(-28/16)R_3 + R_4 \to R_4, -(52/16)R_3 + R_5 \to R_5$. The end result is in echelon form; x, z, w are basic, and y is free; set $y = a$. We solve $-16w = 16$ to get $w = -1$. We solve $z + 3(-1) = -1$ to get $z = 2$. We solve $x + 3y - 3z = -4$ to get $x = 2 - 3a$. Putting it all together, the solution set is $\{(2 - 3a, a, 2, -1)\}$, for any real number a.

4. Write the system as an augmented matrix; put this matrix in row canonical form. Justify each step using elementary row operations. Using the row canonical form, find all solutions to the system (if any).

$$\left[\begin{array}{cccc|c} 1 & 3 & -3 & 0 & -4 \\ 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & -16 & 16 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 3 & -3 & 0 & -4 \\ 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 3 & -3 & 0 & -4 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \to$$

$$\to \left[\begin{array}{cccc|c} 1 & 3 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

We begin where the previous problem left off. The first step is $(-1/16)R_3 \to R_3$. The second step is $-3R_3 + R_2 \to R_2$. The third step is $3R_2 + R_1 \to R_1$. We see immediately that $w = -1, z = -2, x = 2 - 3y$; $y = a$, a free parameter. This yields the same solution as above.

5. Write the system as an augmented matrix; put this matrix in echelon form using partial pivoting. Justify each step using elementary row operations.

$$\left[\begin{array}{cccc|c} 5 & 15 & 0 & -7 & 17 \\ 0 & 0 & 1 & 3 & -1 \\ 3 & 9 & -2 & 5 & -3 \\ 2 & 6 & 3 & -1 & 11 \\ 1 & 3 & -3 & 0 & -4 \end{array}\right] \to \left[\begin{array}{cccc|c} 5 & 15 & 0 & -7 & 17 \\ 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & -2 & 46/5 & -66/5 \\ 0 & 0 & 3 & 9/5 & 21/5 \\ 0 & 0 & -3 & 7/5 & -37/5 \end{array}\right] \to$$

$$\to \left[\begin{array}{cccc|c} 5 & 15 & 0 & -7 & 17 \\ 0 & 0 & 3 & 9/5 & 21/5 \\ 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & -2 & 46/5 & -66/5 \\ 0 & 0 & -3 & 7/5 & -37/5 \end{array}\right] \to \left[\begin{array}{cccc|c} 5 & 15 & 0 & -7 & 17 \\ 0 & 0 & 3 & 9/5 & 21/5 \\ 0 & 0 & 0 & 12/5 & -12/5 \\ 0 & 0 & 0 & 52/5 & -52/5 \\ 0 & 0 & 0 & 16/5 & -16/5 \end{array}\right] \to$$

$$\to \left[\begin{array}{cccc|c} 5 & 15 & 0 & -7 & 17 \\ 0 & 0 & 3 & 9/5 & 21/5 \\ 0 & 0 & 0 & 12/5 & -12/5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Before the first step, we performed $R_1 \leftrightarrow R_5$, to put the 5 in the upper left corner. The next step combines $(-3/5)R_1 + R_3 \to R_3, (-2/5)R_1 + R_4 \to R_4, (-1/5)R_1 + R_5 \to R_5$. The next step was $R_2 \leftrightarrow R_4$, to put the 3 in the upper left corner of the remaining box (it would have also been correct to do $R_2 \leftrightarrow R_5$). The next step combines $(-1/3)R_2 + R_3 \to R_3, (2/3)R_2 + R_4 \to R_4, R_2 + R_5 \to R_5$. The final step combines $(-52/12)R_3 + R_4 \to R_4, (-16/12)R_3 + R_5 \to R_5$.