Math 254 Exam 1 Solutions

1. Carefully state the definition of "basis". Give two examples for the vector space $\mathbb{R}^2$.

   A basis is a set of vectors that is independent, with the added property that any additional vector will make the set dependent. Two examples in $\mathbb{R}^2$ are $\{(1,0),(0,1)\}$ (the standard basis), and $\{(2,2),(3,0)\}$ (this was proved independent in Chapter 0 Problem 6).

2. Let $u = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, and $v = \begin{bmatrix} -1 & 5 & 2 \end{bmatrix}$. For each of the following expressions, determine what type they are (undefined, scalar, matrix/vector).
   (a) uvu, (b) uvv, (c) $(u \cdot v) \cdot v$, (d) $(u \times v) \cdot u$, (e) $(u \times v) \times u$

   u is 3×1. v is 1×3. uv is defined; it is a 3×3 matrix. Therefore, $(uv)u$ is defined, and is a 3×1 matrix (a). However, uv has 3 columns but v has only 1 row, so $(uv)v$ is undefined (b). We may also consider u, v as 3-vectors; in this case, $u \cdot v$ is defined, and is a scalar. However, $(u \cdot v) \cdot v$ is not defined, since dot product is not between a scalar and a vector (c). $u \times v$ is defined, since both u, v are 3-vectors; it is itself a 3-vector. Therefore, $(u \times v) \cdot u$ is defined, since both vectors in the dot product are the same size. Its result is a scalar (d). In addition, $(u \times v) \times u$ is also defined, since both vectors in the (second) cross product are 3-vectors. Its result is a 3-vector (e).

3. For $u = \begin{bmatrix} -3 & 5 & 2 \end{bmatrix}$ and $v = \begin{bmatrix} 4 & 2 & 1 \end{bmatrix}$, determine whether u, v are orthogonal. Justify your answer.

   $u \cdot v = (-3)4 + (5)2 + (2)1 = -12 + 10 + 2 = 0$. Therefore u, v are orthogonal.

4. For $A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 3 \end{bmatrix}$, and $B = \begin{bmatrix} 2 & 1 \\ 0 & -1 \\ 1 & 6 \end{bmatrix}$, calculate AB and BA.

   $$AB = \begin{bmatrix} 0 & -11 \\ 3 & 16 \end{bmatrix}, BA = \begin{bmatrix} 2 & 1 & -1 \\ 0 & -2 & -3 \\ 1 & 12 & 16 \end{bmatrix}$$

5. For $\vec{u} = (1,2,1)$ and $\vec{v} = (-3,-1,0)$, find $\vec{u} \times \vec{v}$ and $\vec{v} \times \vec{u}$.

   $\vec{u} \times \vec{v} = (\hat{\imath} + 2\hat{\jmath} + \hat{k}) \times (-3\hat{\imath} - \hat{\jmath}) = -(\hat{\imath} \times \hat{\jmath}) - 6(\hat{\jmath} \times \hat{\imath}) - 3(\hat{k} \times \hat{\imath}) - (\hat{k} \times \hat{\jmath}) = -\hat{k} - 6(-\hat{k}) - 3\hat{\jmath} - (-\hat{\imath}) = (1, -3, 5)$
   $\vec{v} \times \vec{u} = (-3\hat{\imath} - \hat{\jmath}) \times (\hat{\imath} + 2\hat{\jmath} + \hat{k}) = -6(\hat{\imath} \times \hat{\jmath}) - 3(\hat{\imath} \times \hat{k}) - (\hat{\jmath} \times \hat{\imath}) - (\hat{\jmath} \times \hat{k}) = -6\hat{k} - 3(-\hat{\jmath}) - (-\hat{k}) - \hat{\imath} = (-1, 3, -5)$

   Note: This is no coincidence; it turns out that $(u \times v) = -(v \times u)$ for ALL 3-vectors u, v.