Math 254 Exam 0 Solutions

1. Carefully state the definition of "linear function". Give two examples.

   A linear function consists entirely of some combination of addition and multiplication by constants; furthermore, a nonzero constant may not appear unless it is multiplied by a variable. $f(x) = 3x$, $g(x, y) = 0$.

2. Carefully state the definition of "nondegenerate linear function". Give two examples.

   A linear function is degenerate if it is always equal to zero. All other linear functions are nondegenerate. $f(x) = 3x$, $g(x, y) = -5y$.

3. Consider the vector space $\mathbb{R}^3$. Show that the following set is dependent: $\{(2, 0, 0), (0, 3, 0), (0, 0, 1), (8, 9, 10)\}$.

   We see that $4(2, 0, 0) + 3(0, 3, 0) + 10(0, 0, 1) - 1(8, 9, 10) = (0, 0, 0)$, a nondegenerate linear combination giving the zero vector. ALTERNATE SOLUTION: $\mathbb{R}^3$ has dimension 3, so any four vectors must be linearly dependent.

4. Consider the vector space $\mathbb{R}^3$. Show that $\{(4, 2, 6), (6, 3, 9)\}$ is dependent.

   We see that $3(4, 2, 6) - 2(6, 3, 9) = (12, 6, 18) - (12, 6, 18) = (0, 0, 0)$, a nondegenerate linear combination giving the zero vector.

5. Consider the vector space $\mathbb{R}^3$. Consider the function $f((x_1, x_2, x_3)) = (x_2, x_1 + x_3, 0)$ on this vector space, together with the linear function $g(u, v) = 2u - 3v$. Determine whether or not the composition of f, g can be performed in either order.
   BONUS: Determine whether or not f is a linear transformation (prove your answer).

   $f(g(u, v)) = f(2(u_1, u_2, u_3) - 3(v_1, v_2, v_3)) = f((2u_1 - 3v_1, 2u_2 - 3v_2, 2u_3 - 3v_3)) = (2u_2 - 3v_2, 2u_1 - 3v_1 + 2u_3 - 3v_3, 0)$
   $g(f(u), f(v)) = g((u_2, u_1+u_3, 0), (v_2, v_1+v_3, 0)) = 2(u_2, u_1+u_3, 0) - 3(v_2, v_1+v_3, 0) = (2u_2, 2u_1 + 2u_3, 0) + (-3v_2, -3v_1 - 3v_3, 0) = (2u_2 - 3v_2, 2u_1 + 2u_3 - 3v_1 - 3v_3, 0)$
   These are equal, because each of the three coordinates are equal.
   BONUS: First, we check $g(u, v) = u + v$. $f(g(u, v)) = f((u_1, u_2, u_3) + (v_1, v_2, v_3)) = f((u_1 + v_1, u_2 + v_2, u_3 + v_3)) = (u_2 + v_2, u_1 + v_1 + u_3 + v_3, 0)$. $g(f(u), f(v)) = g((u_2, u_1 + u_3, 0), (v_2, v_1 + v_3, 0)) = (u_2, u_1 + u_3, 0) + (v_2, v_1 + v_3, 0) = (u_2 + v_2, u_1 + u_3 + v_1 + v_3, 0)$. These are equal. Next, we check $g(u) = Au$, where A is any scalar. $f(g(u)) = f(A(u_1, u_2, u_3)) = f((Au_1, Au_2, Au_3)) = (Au_2, Au_1 + Au_3, 0)$. $g(f(u)) = g((u_2, u_1 + u_3, 0)) = A(u_2, u_1 + u_3, 0) = (Au_2, Au_1 + Au_3, 0)$. These are equal. Both required properties hold, so f is in fact a linear transformation.

NOTE: There are many variations to the above solutions. For example, in Problem 4, $-6(4, 2, 6) + 4(6, 3, 9) = (-24, -12, -36) + (24, 12, 36) = (0, 0, 0)$ is a completely different nondegenerate linear combination yielding the zero vector.