MATH 254: Introduction to Linear Algebra

Chapter 0: Fundamental Definitions of Linear Algebra

An equation is called *linear* if it is ENTIRELY some combination of:

1. addition, and
2. multiplication by constants (including 0).

Examples: $3x + 2y = 2$, $2a - b + 0c + 3d = 0$, $3z = 12$, $x_1 + x_2 - x_3 + 4 = 0$
Non-examples: $x^2 = 2$, $xy + y = 7$, $x + \sin(y) = 4$, $e^x = 0$

A function is called *linear* if it ENTIRELY some combination of addition and multiplication by constants, AND no constant is added by itself.
Examples: $f(x, y) = 3x + 2y$, $g(a, b, c, d) = 2a - b + 0c + 3d$, $h(z) = 3z$
Non-examples: $f(x) = x^2$, $f(x, y) = xy + y$, $f(x) = e^x$, $f(x, y) = 3x + 2y + 2$

A linear equation or function is called *nondegenerate* if it contains at least one variable with a coefficient that is NOT zero.
Nondegenerate: $3x + 2y = 2$, $2a - b + 0c + 3d = 0$, $f(z) = 3z$, $g(x_1, x_2) = 7x_1 + 0x_2$
Degenerate: $0x + 0y = 2$, $0a + 0b + 0c + 0d = 0$, $f(x, y) = 0x + 0y$

A *vector space* is a collection of:

1. objects, called vectors, and
2. constants, called scalars. (for us, almost always the real numbers $\mathbb{R}$)

In addition, the vectors and scalars must satisfy a variety of properties (to be named later). The most important property is that any linear function, with any vectors as inputs, must output another vector.
Example 1: $\mathbb{R}^2$ is ordered pairs of real numbers (the vectors). For $u = (1, 2), v = (-1, -1)$, we have $2u + 3v = 2(1, 2) + 3(-1, -1) = (2, 4) + (-3, -3) = (-1, 1)$, which is again a vector.
Example 2: $\mathbb{R}^3$ is ordered triples of real numbers, such as $(1, 2, 3)$.

A collection of vectors, from a vector space, is called *dependent* if there is some nondegenerate linear function on these vectors that gives the zero vector as output. A collection of vectors is called *independent* if EVERY nondegenerate linear function on these vectors gives a nonzero vector as output.
Dependent: $x = (1, 1), y = (1, 2), z = (1, 3)$. The nondegenerate function $f(x, y, z) = x - 2y + z = (1, 1) - 2(1, 2) + (1, 3) = (0, 0)$
Independent: $x = (1, 1), y = (1, 2)$. The function $f(x, y) = 0x + 0y = (0, 0)$, but this is degenerate. It turns out (we will later learn why) that this is the only linear function on $(1, 1), (1, 2)$ that gives the zero vector.

A *basis* of a vector space is an independent set of vectors, and if any other vector is included the result is no longer independent. The number of vectors in a basis is the *dimension* of the vector space.
Example 1: $\mathbb{R}^2$ has dimension 2. $\{(1, 1), (1, 2)\}$ is a basis. So is $\{(1, 0), (0, 1)\}$ (the standard basis), and so is $\{(1, 0), (1, 2)\}$. $\{(1, 0), (2, 0)\}$ is NOT a basis, since it is not independent (because $2(1, 0) - (2, 0) = (0, 0)$). $\{(1, 1)\}$ is NOT a basis, since we could include $(1, 2)$ and it would still be independent.
Example 2: $\mathbb{R}^3$ has dimension 3. The standard basis is $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$.

A one-variable function f on a vector space (or between two vector spaces) is called a *linear transformation* if for any linear function g, they can be performed in any order. For example, for $g(x, y) = 3x + 2y$, $f(g(x, y)) = g(f(x), f(y))$; in other words, $f(3x + 2y) = 3f(x) + 2f(y)$. For f to be a linear transformation, this must work for ANY such g: $f(7x - 2y + 6z) = 7f(x) - 2f(y) + 6f(z)$.
Examples: $f(x) = 2x$, rotation, stretching, matrix multiplication, differentiation
Non-examples: $f(x) = \sin(x)$, because $\sin(\pi/2 + \pi/2) = \sin(\pi) = 0 \neq 2 = \sin(\pi/2) + \sin(\pi/2)$
$f(x) = e^x$, because $e^{0+0} = e^0 = 1 \neq 2 = e^0 + e^0$. $f(x) = x + 3$, because $(7+7)+3 = 17 \neq 20 = (7+3)+(7+3)$

The big idea of Linear Algebra is that every vector (in the right context) is actually a list of scalars. In addition, every linear transformation (in the right context) is actually a matrix multiplication.

Chapter 0 Problems

Solved Problems

1. Carefully state the definition of "Linear Function".

   A linear function is a combination of addition and multiplication by constants ONLY, and no constant is added by itself.

2. Carefully state the definition of "Linear Transformation".

   A linear transformation is a one-variable vector function f such that for ALL linear functions g, their composition may be performed in either order.

3. Determine which of the following equations is linear (justify your answers). A: $0x + 3y = 2y - 7$, B: $0x + 0y + 0z = 7$, C: $3x + 0xy = 7y$, D: $x/y = 3$

   A: This equation has multiplication by constants (0,3,2), and addition/subtraction ONLY. Hence, it is linear. B: This equation has multiplication by constants (0), and addition ONLY. Hence, it is linear. There are no (x, y, z) that satisfy this equation, incidentally. C: The $0xy$ term does not ruin the linearity, because it is equal to zero; this equation is linear. D: The x/y term DOES ruin the linearity. Although it is possible to multiply both sides by y to get $x = 3y$, a linear equation, the two are not the same. $x = y = 0$ satisfies $x = 3y$, but does not satisfy $x/y = 3$, so the two equations are (subtly) different.

4. Consider the vector space $\mathbb{R}^3$, and set $u = (-3, 2, 0), v = (0, 1, 4)$. Calculate $2v - u$.

   $2v - u = 2(-3, 2, 0) - (0, 1, 4) = (-6, 4, 0) + (0, -1, -4) = (-6, 3, -4)$

5. Consider the vector space $\mathbb{R}^2$, and set $u = (1, 1), v = (2, 3), w = (0, 5)$. Determine whether or not $\{u, v, w\}$ is dependent (justify your answer).

   Answer 1: $10u - 5v + w = 10(1, 1) - 5(2, 3) + (0, 5) = (10, 10) - (10, 15) + (0, 5) = (0, 0)$, so $f(u, v, w) = 10u - 5v + w$ is a nondegenerate function on $\{u, v, w\}$ that gives zero as output. Hence, $\{u, v, w\}$ is dependent.
   Answer 2: If $\{u, v, w\}$ were independent, then there would be a basis of $\mathbb{R}^2$ with at least three vectors (either this set is a basis, or we can keep adding vectors until we get a basis). However, $\mathbb{R}^2$ has dimension 2, so this is impossible. Therefore, $\{u, v, w\}$ must be dependent.

6. Consider the vector space $\mathbb{R}^2$, and set $u = (2, 2), v = (3, 0)$. Determine whether or not $\{u, v\}$ is dependent (justify your answer).

   Suppose that $\{u, v\}$ were dependent. Then, there is some nondegenerate linear function giving the zero vector as output. That is, there are some constants a, b (not both zero) so that $au + bv = (0, 0)$. We calculate $au + bv = a(2, 2) + b(3, 0) = (2a, 2a) + (3b, 0) = (2a + 3b, 2a) = (0, 0)$. So, we must have $2a + 3b = 0$ and $2a = 0$. The second equation gives us $a = 0$; we plug that into the first equation and get $b = 0$. Hence, $a = b = 0$ and the linear function was actually degenerate. Therefore, there is no nondegenerate linear function giving the zero vector, and therefore $\{u, v\}$ is independent.

7. Consider the vector space $\mathbb{R}^2$ and the vector function $f((x_1, x_2)) = (2x_2, x_1)$. For the function $g(u, v) = 3u - 7v$, determine whether or not the composition of f, g can be performed in either order. What does this tell you about f?

   Let $u = (u_1, u_2), v = (v_1, v_2)$. If we perform g first, then f, then we get $f(g(u, v)) = f(3(u_1, u_2) - 7(v_1, v_2)) = f((3u_1, 3u_2) + (-7v_1, -7v_2)) = f((3u_1 - 7v_1, 3u_2 - 7v_2)) = (2(3u_2 - 7v_2), 3u_1 - 7v_1) = (6u_2 - 14v_2, 3u_1 - 7v_1)$. On the other hand, if we perform f first, then g, then we get $g(f(u), f(v)) = g(f((u_1, u_2)), f((v_1, v_2))) = g((2u_2, u_1), (2v_2, v_1)) = 3(2u_2, u_1) - 7(2v_2, v_1) = (6u_2, 3u_1) + (-14v_2, -7v_1) = (6u_2 - 14v_2, 3u_1 - 7v_1)$. This is the same as before. This calculation does not prove that f is a linear transformation – to prove this we would need to test ALL linear functions g, not just this one. This calculation doesn't prove that

f isn't a linear transformation, either, since the two results are the same. This calculation lends evidence that f might be a linear transformation, but is not conclusive.

8. Consider the vector space $\mathbb{R}^2$ and the vector function $f((x_1, x_2)) = (x_1x_2, 0)$. For the function $g(u, v) = 3u - 7v$, determine whether or not the composition of f, g can be performed in either order. What does this tell you about f?

   Let $u = (u_1, u_2), v = (v_1, v_2)$. If we perform g first, then f, then we get $f(g(u, v)) = f(3(u_1, u_2) - 7(v_1, v_2)) = f((3u_1, 3u_2) + (-7v_1, -7v_2)) = f((3u_1 - 7v_1, 3u_2 - 7v_2)) = ((3u_1 - 7v_1)(3u_2 - 7v_2), 0)$. On the other hand, if we perform f first, then g, then we get $g(f(u), f(v)) = g(f((u_1, u_2)), f((v_1, v_2))) = g((u_1u_2, 0), (v_1v_2, 0)) = 3(u_1u_2, 0) - 7(v_1v_2, 0) = (3u_1u_2 - 7v_1v_2, 0)$. Are these equal? If so, we would have the first coordinates equal: $(3u_1 - 7v_1)(3u_2 - 7v_2) = 3u_1u_2 - 7v_1v_2$. The two sides LOOK different, but appearances can be deceiving. We need to find out if they are really different, or not. One good strategy, before diving into algebra, is to try plugging in some values. Let's try something simple: $u_1 = u_2 = v_1 = v_2 = 0$. Both sides are zero, so maybe they are equal. Let's try again: $u_1 = u_2 = v_1 = v_2 = 1$. The left side is $(3 - 7)(3 - 7) = 16$. The right side is $3 - 7 = -4$. Aha, the two sides are not equal, hence the composition of f, g may NOT be performed in either order. We conclude that f is NOT a linear transformation.

Supplementary Problems

9. Carefully state the definition of "Linear Equation".
10. Carefully state the definition of "Basis".
11. Determine which of the following functions is linear (justify your answers). A: $f(x, y) = 7x - 3y + 2x + 4y$, B: $f(x, y) = 0x + 0y + 0$, C: $f(x, y) = 2x + 3y + 4$, D: $f(x, y, z) = (x/y)(y/z)z$, E: $f(x, y, z) = x$
12. Consider the vector space $\mathbb{R}^2$, and set $u = (2, 6), v = (-3, -9)$. Determine whether or not $\{u, v\}$ is independent (justify your answer).
13. Consider the vector space $\mathbb{R}^2$, and set $u = (2, 6), v = (0, -9)$. Determine whether or not $\{u, v\}$ is independent (justify your answer).
14. Consider the vector space $\mathbb{R}^2$ and the vector function $f((x_1, x_2)) = (x_2, 0)$. For the function $g(u, v) = 3u - 7v$, determine whether or not the composition of f, g can be performed in either order. What does this tell you about f?
15. Consider the vector space $\mathbb{R}^2$ and the vector function $f((x_1, x_2)) = (x_1^3, 0)$. For the function $g(u, v) = 3u - 7v$, determine whether or not the composition of f, g can be performed in either order. What does this tell you about f?

Solutions

11: A,B,E   12: no   13: yes   14: yes   15: no