Mathematical Maturity via Discrete Mathematics

Big Book O' Exercises*

Vadim Ponomarenko

Spring 2023 Edition

*Also hints, starting on page 36.

Exercises for Chapter 1.

Exercises for Chapter 1.1.

1.1. *Find three definitions from later chapters in this textbook. Copy the definitions carefully, and give the page on which they appear. Indicate the context (if any), category, and verb.*

1.2. *Carefully write definitions for the following terms. Underline the category and verb in each.* [Hint 229]

a. *pair of consecutive integers*

b. *perfect square*

c. *perfect cube*

d. *perfect power*

e. *purely imaginary number*

1.3. *Find a mathematical definition from any other published source, for a term that does not appear in this text. Copy the definition carefully, and give the source where you found it. Indicate the context (if any), category, and verb.*

1.4. *Let's say that a curve in the plane is* even+ *if there is some vertical line of symmetry*[1]*. That is, we can fold the paper on that line and the two halves of the curve will exactly coincide. Prove that* $y = x^2, y = (x+3)^2 + 5, y = 7$ *are all even+. Prove that* $y = x^3$ *is not even+.* [Hint 206]

1.5. *Let's say that a curve in the plane is* odd+ *if there is some point of rotational symmetry*[2]*. That is, we can rotate the paper 180 degrees*

[1] not necessarily the y-axis
[2] not necessarily the origin

at that point and the curve pre-rotation will exactly coincide with the curve post-rotation. Prove that $y = x, y = x^3, y = (x+3)^3 + 5$ *are all odd+. Prove that* $y = x^2$ *is not odd+.* *[Hint 295]*

1.6. *Prove that* $y = \sin x$ *is both even+ and odd+.* *[Hint 41]*

Exercises for Chapter 1.2.

1.7. *Prove that 6 is even and 7 is odd.* *[Hint 99]*

1.8. *Apply Theorem 1.5 to* $a = -100, b = 3$. *(i.e. find* q, r*)* *[Hint 143]*

1.9. *Let* a, b *be odd. Prove that* $a + b$ *is even.* *[Hint 224]*

1.10. *Let* a, b *be odd. Prove that* ab *is odd.*

1.11. *Let* a *be even, and let* b, c *be odd. Prove that* $ab + ac + bc$ *is odd.* *[Hint 184]*

1.12. *Prove Theorem 1.7, by assuming* n *that is both odd and even, and deriving a contradiction.* *[Hint 273]*

Exercises for Chapter 1.3.

1.13. *Prove the unproved parts of Theorem 1.10.* *[Hint 279]*

1.14. *Prove the unproved parts of Theorem 1.11.* *[Hint 275]*

1.15. *Let* $a, b, c, d \in \mathbb{Z}$. *Suppose that* $a \leq b < c$. *Prove that* $a + d \leq b + d < c + d$. *[Hint 234]*

1.16. *Let* $a, b, c, a', b', c' \in \mathbb{Z}$. *Suppose that* $a < b \leq c$ *and* $a' < b' < c'$. *Prove that* $a + a' < b + b' < c + c'$. *[Hint 201]*

1.17. *Let* $a, b \in \mathbb{Z}$. *Suppose that* $0 \leq a \leq b$. *Prove that* $0 \leq a^2 \leq b^2$.

1.18. *Prove the unproved parts of Theorem 1.12.*

1.19. *Calculate* $\lceil \lceil \pi \rceil \lceil \pi \rceil \rceil - \lceil \pi^2 \rceil$. *[Hint 178]*

1.20. *Find* $x, y \in \mathbb{R}$ *such that* $x < y < 0$ *but* $\lceil x \rceil > \lfloor y \rfloor$.

1.21. *Suppose that* $x \in \mathbb{R}$. *Prove that if* $\lfloor x \rfloor = \lceil x \rceil$, *then* $x \in \mathbb{Z}$. *[Hint 77]*

1.22. *Suppose that* $a|b$ *and* $c \in \mathbb{Z}$. *Prove that* $a|(bc)$.

1.23. *Suppose that* $a|b$ *and* $b|c$. *Prove that* $a|c$.

1.24. *Suppose that* $a|b$ *and* $a|c$. *Prove that* $a|(b+c)$.

1.25. *For each of the following numbers, classify as prime, composite, both, or neither:* $6, 5, \pi, 1, 0, -1, -5, -6$. *Be sure to justify your answers. [Hint 13]*

1.26. *Suppose that* p *is prime. Prove that* p^2 *is composite.*

1.27. *Calculate* $\frac{(\lceil 9.9 \rceil)!}{(\lfloor 9.9 \rfloor)!}$. *[Hint 177]*

1.28. *For arbitrary* $n \in \mathbb{N}$, *calculate and simplify* $\frac{(n+2)!}{n!}$. *[Hint 236]*

1.29. *Let* $a, b \in \mathbb{N}_0$ *with* $a \geq b$. *Prove that* $\binom{a}{0} = \binom{a}{a} = 1$, *and that* $\binom{a}{b} = \binom{a}{a-b}$. *[Hint 94]*

1.30. *Let* $a, b \in \mathbb{N}_0$ *with* $a > b$. *Prove that* $\binom{a}{b} + \binom{a}{b+1} = \binom{a+1}{b+1}$. *[Hint 37]*

Exercises for Chapter 2.

Exercises for Chapter 2.1.

The following three exercises should be done by cases (not truth tables).

2.1. *Let* p, q *be propositions. Prove that* $p \wedge q \wedge (\neg p) \equiv F$. *[Hint 274]*

2.2. *Prove Theorem 2.7. (6 parts) [Hint 122]*

2.3. *Prove Theorem 2.8. (2 parts) [Hint 231]*

Exercises for Chapter 2.2.

The remaining proofs in Chapter 2 should be done with truth tables.

2.4. *Prove the unproved half of Theorem 2.10.*

2.5. *Prove Theorem 2.11. (2 parts)*

2.6. *Prove Theorem 2.12. (2 parts)*

2.7. *Simplify* $\neg((p \wedge q) \vee ((r \vee \neg q) \wedge \neg s))$ *as much as possible (i.e. where only basic propositions are negated). [Hint 87]*

2.8. *Prove or disprove that* $(p \vee q) \wedge r \equiv p \vee (q \wedge r)$. *[Hint 237]*

Exercises for Chapter 2.3.

2.9. *Rewrite* $(p \to q) \to (r \to s)$ *to use only* $\neg, \vee, \wedge$. *[Hint 158]*

2.10. *Simplify* $\neg((p \wedge q) \vee (r \to s))$ *as much as possible (i.e. where only basic propositions are negated).*

2.11. *Simplify* $\neg(((p \to q) \to r) \wedge s)$ *as much as possible (i.e. where only basic propositions are negated).*

2.12. *Let* p, q *be propositions. Prove that* $p \oplus q \equiv (p \wedge (\neg q)) \vee ((\neg p) \wedge q)$. *[Hint 80]*

2.13. *Let* p, q *be propositions. Prove that* $p \leftrightarrow q \equiv (p \wedge q) \vee ((\neg p) \wedge (\neg q))$.

2.14. *Let* p, q *be propositions. Prove that* $p \uparrow q \equiv \neg(p \wedge q)$.

2.15. *Let* p, q *be propositions. Prove that* $p \downarrow q \equiv \neg(p \vee q)$.

2.16. *Let* p, q, r *be propositions. Prove that* $(p \wedge q) \vee (\neg r)$ *is not equivalent to* $q \vee (r \rightarrow p)$. *[Hint 98]*

2.17. *Prove Theorem 2.17. (2 parts) [Hint 23]*

2.18. *Let* p, q *be propositions. Prove each of the following equivalences.*

a. $\neg p \equiv p \uparrow p$,

b. $p \wedge q \equiv (p \uparrow q) \uparrow (p \uparrow q)$, *and*

c. $p \vee q \equiv (p \uparrow p) \uparrow (q \uparrow q)$.

These equivalences show that $\uparrow$ *is "universal"; i.e. all other operators can be built using just* $\uparrow$.

2.19. *Let* p, q *be propositions. Prove each of the following equivalences.*

a. $\neg p \equiv p \downarrow p$,

b. $p \wedge q \equiv (p \downarrow p) \downarrow (q \downarrow q)$, *and*

c. $p \vee q \equiv (p \downarrow q) \downarrow (p \downarrow q)$.

These equivalences show that $\downarrow$ *is "universal"; i.e. all other operators can be built using just* $\downarrow$.

Exercises for Chapter 3.

Exercises for Chapter 3.1.

3.1. *Prove the unproved part of Theorem 3.2.* [Hint 284]

3.2. *Let* p, q, r *be propositions. Use a truth table to prove the "Hypothetical Syllogism" Theorem:* $(p \to q), (q \to r) \vdash (p \to r)$.

3.3. *Let* p, q, r *be propositions. Use a truth table to prove the "Composition" Theorem:* $(p \to q) \wedge (p \to r) \vdash (p \to (q \wedge r))$.

3.4. *Let* p, q, r, s *be propositions. Use a truth table to prove* $p \to q, q \to r, r \to s, p \vdash s$. [Hint 297]

3.5. *Let* p, q, r, s *be propositions. Use a truth table to prove the "Constructive Dilemma" Theorem:* $(p \to q), (r \to s), (p \vee r) \vdash (q \vee s)$. [Hint 9]

Exercises for Chapter 3.2.

3.6. *Use truth tables to prove Theorem 3.5. (6 parts)* [Hint 162]

A moment of silence please, for the era of truth tables is over. For all of the remaining exercises in the book, **you may no longer use truth tables**. For the rest of chapter 3, you should use semantic theorems, proof by cases, and the proof methods from section 3.3.

3.7. *Prove modus ponens (Theorem 3.3) using conditional interpretation (Theorem 2.15).* [Hint 3]

3.8. *Let* p, q, r, s *be propositions. Prove* $p \to q, q \to r, r \to s, p \vdash s$ *without truth tables.* [Hint 32]

3.9. *Prove modus tollens (Theorem 3.5.a.) using conditional interpretation (Theorem 2.15).* [Hint 152]

3.10. *Prove disjunctive syllogism (Theorem 3.5.e.) using conditional interpretation and double negation.* *[Hint 238]*

3.11. *Let* p, q, r, s *be propositions. Prove the "Constructive Dilemma" Theorem:* $(p \to q), (r \to s), (p \vee r) \vdash (q \vee s)$. *[Hint 34]*

3.12. *Let* p, q, r, s *be propositions. Prove the "Destructive Dilemma" Theorem:* $(p \to q), (r \to s), ((\neg q) \vee \neg s) \vdash ((\neg p) \vee \neg r)$. *[Hint 250]*

3.13. *Let* p, q, r *be propositions. Prove the "Composition" Theorem:* $(p \to q) \wedge (p \to r) \vdash (p \to (q \wedge r))$. *[Hint 6]*

3.14. *Let* p, q, r *be propositions. Prove the "Hypothetical Syllogism" Theorem:* $(p \to q), (q \to r) \vdash (p \to r)$. *[Hint 276]*

3.15. *Let* p, q, r, s *be propositions. Suppose that* $(p \wedge q) \vee r$, $\mathsf{T} \to s$, *and* $r \to \neg s$ *are all* T. *Prove that* p *must be* T.

Exercises for Chapter 3.3.

3.16. *Let* $x \in \mathbb{R}$. *Prove that if* 2 *is irrational, then* x *is rational.* *[Hint 10]*

3.17. *Let* $x \in \mathbb{R}$. *Prove that if* x *is rational, then* 2 *is rational.* *[Hint 221]*

3.18. *Let* $x \in \mathbb{R}$. *Prove that if* x *is rational, then* $2+x$ *is rational.* *[Hint 259]*

3.19. *Let* $x \in \mathbb{R}$. *Prove that if* x *is irrational, then* $2+x$ *is irrational.* *[Hint 241]*

3.20. *Compute the converse of the converse of conditional proposition* $p \to q$. *[Hint 211]*

3.21. *Compute the contrapositive of the contrapositive of conditional proposition* $p \to q$. *[Hint 95]*

3.22. *Compute the contrapositive of the inverse of the converse of conditional proposition* $p \to q$. *[Hint 133]*

3.23. *Prove the unproved part of Theorem 3.13.* *[Hint 188]*

3.24. *Prove Theorems 3.14 and 3.15. Note: although the theorems look similar, the proofs need to be very different.* *[Hint 289]*

Exercises for Chapter 4.

Exercises for Chapter 4.1.

4.1. *Classify each of the following statements as proposition, predicate, or not well-formed:*

a. $(x > 6) \wedge (y > x)$.

b. $\exists x \in \mathbb{Z},\ \exists y \in \mathbb{Z},\ (x > 6) \wedge (y > x)$.

c. $\exists x \in \mathbb{Z},\ (x > 6) \wedge (\exists y \in \mathbb{Z},\ y > x)$.

d. $\exists y \in \mathbb{Z},\ (x > 6) \wedge (\exists x \in \mathbb{Z},\ y > x)$.

e. $\exists x \in \mathbb{Z},\ (x > 6) \wedge (y > x)$.

f. $\exists x \in \mathbb{R},\ \exists y \in \mathbb{Z},\ x + y = z$.

g. $\exists x \in \mathbb{R},\ \exists y \in \mathbb{Z},\ x + y = z,\ \exists z \in \mathbb{Z}$.

h. $\exists x \in \mathbb{R},\ \exists y \in \mathbb{Z},\ (x < y) \wedge (\exists z \in \mathbb{Z},\ x + y = z)$.

[Hint 66]

4.2. *Find a well-formed expression with one bound and two free variables.* *[Hint 137]*

Exercises for Chapter 4.2.

4.3. *Prove that* $\forall x \in \mathbb{N},\ 2x + 5 \geq 7$. *[Hint 29]*

4.4. *Disprove that* $\forall x \in \mathbb{N},\ 2x + 5 \geq 10$. *[Hint 153]*

4.5. *Prove or disprove that* $\forall x \in \mathbb{N},\ x^2 - 4x + 5 \geq 0$. *[Hint 219]*

4.6. *Prove or disprove that* $\forall x \in \mathbb{N},\ x^2 - 6x + 5 \geq 0$. *[Hint 252]*

4.7. *Prove that* $\exists x \in \mathbb{N},\ |3x - 5| \leq 1$. *[Hint 107]*

4.8. *Disprove that* $\exists x \in \mathbb{N},\ |3x-5| < 1$. *[Hint 296]*

4.9. *Prove or disprove that* $\exists x \in \mathbb{N},\ |3x-8| = 1$. *[Hint 198]*

4.10. *Prove or disprove that* $\exists x \in \mathbb{N},\ |3x-8| = 3$. *[Hint 287]*

Exercises for Chapter 4.3.

4.11. *Let* p *denote "*$\forall x,\ \forall y,\ \exists z,\ P(x,y,z)$*", and let* q *denote the proposition "*$\forall x,\ \exists z,\ \forall y,\ P(x,y,z)$*". Is one of these propositions stronger than the other?* *[Hint 213]*

4.12. *Let* p *denote "*$\forall x,\ \exists z,\ \forall y,\ P(x,y,z)$*", and let* q *denote the proposition "*$\exists z,\ \forall x,\ \forall y,\ P(x,y,z)$*". Is one of these propositions stronger than the other?* *[Hint 245]*

4.13. *Consider the two doubly quantified propositions* p*="to every thing there is a season", and* q*="there is a season to every thing". Is one of these propositions stronger than the other?*

4.14. *Simplify the proposition* $\neg(\exists x,\ \exists y,\ \forall z,\ x+y=z)$ *as much as possible (no quantifiers or compound propositions should be negated).* *[Hint 269]*

4.15. *Simplify the proposition* $\neg(\forall x,\ \exists y,\ \forall z,\ x \le y < z)$ *as much as possible (no quantifiers or compound propositions should be negated).* *[Hint 148]*

4.16. *Simplify the proposition* $\neg(\exists x,\ \forall y,\ \forall z,\ (x<y) \to (x<z))$ *as much as possible (no quantifiers or compound propositions should be negated).* *[Hint 136]*

4.17. *Simplify the proposition* $\neg(\exists x,\ \forall y,\ \forall z,\ (x<y) \leftrightarrow (x<z))$ *as much as possible (no quantifiers or compound propositions should be negated).* *[Hint 144]*

4.18. *Prove the proposition* $\forall x \in \mathbb{R},\ \forall y \in \mathbb{R},\ \exists z \in \mathbb{R}, (x-z)^2 \le (x-y)^2$. *[Hint 205]*

4.19. *Prove the proposition* $\forall x \in \mathbb{R},\ \exists y, z \in \mathbb{R},\ y^2 \le x^2 < z^2$. *[Hint 242]*

4.20. *Disprove the proposition* $\forall x \in \mathbb{N},\ \exists y \in \mathbb{N}, y < x$. *[Hint 246]*

4.21. *Disprove the proposition* $\exists x \in \mathbb{N},\ \forall y \in \mathbb{N}, y < x$. *[Hint 138]*

4.22. *Prove or disprove the proposition* $\forall x \in \mathbb{N},\ \exists y \in \mathbb{N},\ x^2 < y < (x+1)^2$. *[Hint 89]*

4.23. *Prove or disprove the proposition* $\forall x \in \mathbb{Z},\ \forall y \in \mathbb{Z},\ \exists z \in \mathbb{Z},\ (x < y) \to (x < z < y)$. *[Hint 44]*

4.24. *Prove or disprove the proposition* $\forall x \in \mathbb{R},\ \forall y \in \mathbb{R},\ \exists z \in \mathbb{R},\ (x < y) \to (x < z < y)$. *[Hint 232]*

Exercises for Chapter 5.

Exercises for Chapter 5.1.

5.1. *Let a, b be odd. Use proof by contradiction to prove that $a+b$ is even.* [Hint 112]

5.2. *Let a be irrational. Use proof by contradiction to prove that $a+2$ is irrational.* [Hint 200]

5.3. *Let $n \in \mathbb{Z}$. Prove that $\frac{n(n-3)}{2}$ is an integer.* [Hint 132]

5.4. *Let $n \in \mathbb{Z}$. Prove that $\frac{n^2-3n+2}{2}$ is an integer.* [Hint 261]

5.5. *Let $n \in \mathbb{Z}$. Use the Division Algorithm (Theorem 1.5) with $b = 3$ to prove that $\frac{n(n-1)(n-2)}{3}$ is an integer.* [Hint 155]

5.6. *Let $x \in \mathbb{R}$. Use cases to prove that $|x-1| + |x+1| \geq 2$.* [Hint 167]

5.7. *Let $x \in \mathbb{R}$. Use cases to prove that $|x-1| + |x+1| + |x| \geq 2$.* [Hint 262]

5.8. *Prove that $\sqrt{3}$ is irrational.* [Hint 182]

Exercises for Chapter 5.2.

5.9. *Re-prove Example 5.10 using the proof structure $\{a \vdash c, c \vdash b, b \vdash d, d \vdash a\}$.* [Hint 46]

5.10. *Give five different proof structures, any of which would prove Example 5.10. You do not need to actually write all these proofs, just state the proof structures.* [Hint 258]

5.11. *Let $w, x, y, z \in \mathbb{N}$. Prove that the following are equivalent: (a) $\frac{w}{x} = \frac{y}{z}$; (b) $\frac{wz-xy}{xz} = 0$; (c) $wz - xy = 0$; (d) $\frac{w}{y} = \frac{x}{z}$.* [Hint 17]

5.12. *Let $n \in \mathbb{Z}$ be even. Prove that there is a unique $m \in \mathbb{Z}$ with $n = 2m$, i.e. $\exists! m \in \mathbb{Z},\ n = 2m$.* [Hint 253]

5.13. *Prove or disprove that* $!n \in \mathbb{N},\ |2n-1| = 3$. *[Hint 56]*

5.14. *Prove or disprove that* $!n \in \mathbb{Z},\ |2n-2| = 3$. *[Hint 58]*

5.15. *Prove or disprove that* $!n \in \mathbb{Z},\ |2n-2| = 4$. *[Hint 50]*

5.16. *Let* $n \in \mathbb{N}$. *Prove* $!m \in \mathbb{N},\ n = m(m+1)$. *[Hint 105]*

Exercises for Chapter 5.3.

5.17. *Let* $x \in \mathbb{R}$. *Prove that* $\lfloor x \rfloor \leq \lceil x \rceil$. *[Hint 142]*

5.18. *Let* $x \in \mathbb{R}$. *Prove that* $\lfloor \lfloor x \rfloor \rfloor = \lfloor x \rfloor$. *[Hint 170]*

5.19. *Let* $n \in \mathbb{Z}$. *Prove that* $\lfloor \frac{n}{2} \rfloor \geq \frac{n-1}{2}$. *[Hint 244]*

5.20. *Let* $x \in \mathbb{R}$. *Prove that* $\lfloor -x \rfloor = -\lceil x \rceil$. *[Hint 55]*

5.21. *Prove the unproved part of Theorem 5.17.* *[Hint 109]*

5.22. *Prove the unproved part of Theorem 5.18.* *[Hint 115]*

5.23. *Prove the unproved part of Theorem 5.19.* *[Hint 173]*

5.24. *Prove the unproved part of Theorem 5.20.* *[Hint 223]*

5.25. *Let* $x \in \mathbb{R}$. *Prove that* $2\lfloor x \rfloor \leq \lfloor 2x \rfloor \leq 2\lfloor x \rfloor + 1$. *[Hint 150]*

5.26. *Let* $x \in \mathbb{R}$. *Prove that* $\lfloor x + \frac{1}{2} \rfloor = \lfloor x \rfloor$ *if and only if* $x - \lfloor x \rfloor < \frac{1}{2}$. *[Hint 196]*

Exercises for Chapter 6.

All of the Chapter 6 exercises should be done with some form of induction.

Exercises for Chapter 6.1.

6.1. *Prove that, for every* $n \in \mathbb{N}$, $3^n > 2^n$. *[Hint 256]*

6.2. *Prove that, for every* $n \in \mathbb{N}$, $10^n > n^2$. *[Hint 40]*

6.3. *Prove that, for every* $n \in \mathbb{N}$, $\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$. *[Hint 175]*

6.4. *Prove that, for every* $n \in \mathbb{N}$, $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$. *[Hint 168]*

6.5. *Prove that, for every* $n \in \mathbb{N}$, $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$.

6.6. *Prove that, for every* $n \in \mathbb{N}$, $\sum_{i=1}^{n}(-1)^i i^2 = \frac{n(n+1)(-1)^n}{2}$. *[Hint 260]*

6.7. *Prove that, for every* $n \in \mathbb{N}$, $\frac{(2n)!}{n!n!} \geq \frac{4^n}{2n+1}$. *[Hint 282]*

6.8. *Prove that, for every* $n \in \mathbb{N}$, $\sum_{i=0}^{n}(2i+1) = (n+1)^2$. *[Hint 28]*

Exercises for Chapter 6.2.

6.9. *Prove that, for every* $n \in \mathbb{N}_0$, $2^n > n$. *[Hint 171]*

6.10. *Prove that, for every* $n \in \mathbb{N}$ *with* $n \geq 2$, $n^3 \geq 2n + 1$. *[Hint 11]*

6.11. *Prove that, for every* $n \in \mathbb{N}$ *with* $n \geq 4$, $n! > n^2$. *[Hint 291]*

6.12. *(Bernoulli's inequality) Let* $x \in \mathbb{R}$ *with* $x > -1$. *Prove that, for every* $n \in \mathbb{N}_0$, $(1+x)^n \geq 1 + nx$. *[Hint 12]*

6.13. *(sum of arithmetic series) Let* $a, d \in \mathbb{R}$. *Prove that, for every* $n \in \mathbb{N}_0$, $\sum_{i=0}^{n}(a + id) = \frac{n+1}{2}(2a + nd)$. *[Hint 26]*

6.14. *(sum of geometric series) Let* $a, r \in \mathbb{R}$ *with* $r \neq 1$. *Prove that, for every* $n \in \mathbb{N}_0$, $\sum_{i=0}^{n} ar^i = a\frac{1-r^{n+1}}{1-r}$. *[Hint 249]*

6.15. *Prove that, for every* $n \in \mathbb{N}_0$*, the Fibonacci numbers satisfy* $F_{n+2} = 1 + \sum_{i=0}^{n} F_i$. *[Hint 83]*

6.16. *Prove that, for every* $n \in \mathbb{N}_0$*, the Fibonacci numbers satisfy* $F_n F_{n+1} = \sum_{i=0}^{n} F_i^2$. *[Hint 92]*

6.17. *Prove that, for every* $n \in \mathbb{N}$*, the Fibonacci numbers satisfy* $F_{2n+1} = 1 + \sum_{i=0}^{n} F_{2i}$. *[Hint 270]*

6.18. *Prove that, for every* $n \in \mathbb{N}$*, the Fibonacci numbers satisfy* $F_{2n} = \sum_{i=0}^{n-1} F_{2i+1}$. *[Hint 290]*

6.19. *(Cassini's identity) Prove that, for every* $n \in \mathbb{N}$ *with* $n \geq 2$*, the Fibonacci numbers satisfy* $F_{n-1}^2 - F_n F_{n-2} = (-1)^n$. *[Hint 14]*

6.20. *Prove that, for every* $n \in \mathbb{N}$ *with* $n \geq 11$*, the Fibonacci numbers satisfy* $F_n \geq 1.5^n$*. This is a counterpoint to Theorem 6.13.* *[Hint 118]*

Exercises for Chapter 6.3.

6.21. *Let* $x \in \mathbb{R}$*. Use minimum element induction to prove that there is a unique* $n \in \mathbb{Z}$ *such that* $n - 1 < x \leq n$*. This proves that the ceiling function is well-defined.* *[Hint 191]*

6.22. *Let* $x \in \mathbb{R}$*. Use minimum element induction to prove that there is a unique* $n \in \mathbb{Z}$ *such that* $n \leq x < n + 1$*. Note: this is exactly the statement of Thm 6.17, but proved with minimum element induction instead of maximum element induction.* *[Hint 1]*

6.23. *Let* $a, b \in \mathbb{Z}$ *with* $b \geq 1$*. Use minimum element induction to prove that there are* $q, r \in \mathbb{Z}$ *satisfying* $a = bq + r$ *and* $0 < r \leq b$*.* *[Hint 166]*

6.24. *Let* $a, b \in \mathbb{Z}$ *with* $b \geq 1$*. Use minimum element induction to prove that there are* $q, r \in \mathbb{Z}$ *satisfying* $a = bq + r$ *and* $0 \leq r < b$*. Note: this is exactly the statement of Thm 6.18, but proved with minimum element induction instead of maximum element induction.* *[Hint 271]*

6.25. *Let* $a, b \in \mathbb{Z}$ *with* $b \geq 1$. *Use maximum element induction to prove that there are* $q, r \in \mathbb{Z}$ *satisfying* $a = bq + r$ *and* $-1 \leq r < b - 1$. *[Hint 106]*

6.26. *Find a formula for the* n^{th} *smallest element in* $\mathbb{Z}$*, by the non-standard order* $\prec$ *given by* $0 \prec 1 \prec -1 \prec 2 \prec -2 \prec 3 \prec -3 \prec \cdots$. *The smallest is* 0*, the second smallest is* 1*, the* n^{th} *smallest is ...?* *[Hint 141]*

6.27. *Find a non-standard ordering* $\prec$ *so that the positive rationals* $\{\frac{a}{b} : a, b \in \mathbb{N}\}$ *are well-ordered by* $\prec$. *[Hint 7]*

Exercises for Chapter 7.

Exercises for Chapter 7.1.

7.1. *Determine the order, if any, of each of the following recurrence relations.* *(a)* $a_n = 7a_{n-2}$; *(b)* $b_n = b_{n-1} - 3b_{n-3} + b_{n-2}$; *(c)* $c_n = \sum_{i=1}^{n-1} 2^i c_i$; *(d)* $d_n = 7 + d_{n-2}$; *(e)* $e_n = 2e_{n-2} - 4e_{n-1}$; *(f)* $f_n = 3$. [Hint 113]

7.2. *Consider the recurrence with initial conditions* $a_0 = 1, a_1 = 1$, *and relation* $a_n = 5a_{n-1} - 6a_{n-2}$ $(n \geq 2)$. *Compute* a_5 *by iteratively applying the relation to the initial conditions.* [Hint 272]

7.3. *Solve the recurrence with initial conditions* $a_0 = 5, a_1 = 12$, *and relation* $a_n = 5a_{n-1} - 6a_{n-2}$ $(n \geq 2)$. [Hint 102]

7.4. *Solve the recurrence with initial conditions* $a_0 = 1, a_1 = 1$, *and relation* $a_n = 5a_{n-1} - 6a_{n-2}$ $(n \geq 2)$.

7.5. *Solve the recurrence with initial conditions* $a_0 = 3, a_1 = -16$, *and relation* $a_n = a_{n-1} + 6a_{n-2}$ $(n \geq 2)$. [Hint 125]

7.6. *Solve the recurrence with initial conditions* $a_0 = 1, a_1 = -9$, *and relation* $a_n = 9a_{n-2}$ $(n \geq 2)$. [Hint 292]

7.7. *Solve the recurrence with initial conditions* $a_0 = 3, a_1 = 4$, *and relation* $a_n = 4a_{n-1} - 4a_{n-2}$ $(n \geq 2)$. [Hint 31]

7.8. *Solve the recurrence with initial conditions* $a_0 = 0, a_1 = 10$, *and relation* $a_n = 4a_{n-1} - 4a_{n-2}$ $(n \geq 2)$.

7.9. *Solve the recurrence with initial conditions* $a_0 = 2, a_1 = 5$, *and relation* $a_n = 2a_{n-1} - a_{n-2}$ $(n \geq 2)$.

7.10. *Solve the Fibonacci recurrence. It has initial conditions* $F_0 = 0, F_1 = 1$, *and relation* $F_n = F_{n-1} + F_{n-2}$ $(n \geq 2)$. *Hint: The roots are irrational.* [Hint 68]

7.11. *Carefully adapt the second-order algorithm for first-order recurrences. You will only have one initial condition, one root, and one parameter.* *[Hint 145]*

7.12. *Solve the recurrence with initial condition* $a_0 = 5$*, and relation* $a_n = 3a_{n-1}$ *(*$n \geq 1$*).* *[Hint 169]*

Exercises for Chapter 7.2.

7.13. *Let* $a_n = 1,000,000n + 3,000,000$*. Prove that* $a_n = O(n)$*.* *[Hint 216]*

7.14. *Let* $a_n = 5 + \frac{1}{n} + \frac{1}{n+1}$*. Prove that* $a_n = O(1)$*.*
Note: here $\lim_{n\to\infty} a_n = 5$*, rather than the more common* ∞*.* *[Hint 176]*

7.15. *Let* $a_n = n^2 + n + 1 + \frac{1}{n} + \sin n$*. Prove that* $a_n = O(n^2)$*.* *[Hint 62]*

7.16. *Let* $a_n = n^{2.1}$*. Without using the classification theorem, prove that* $a_n \neq O(n^2)$*.* *[Hint 154]*

7.17. *Let* $a_n = 2.1^n$*. Without using the classification theorem, prove that* $a_n \neq O(2^n)$*.* *[Hint 163]*

7.18. *Let* $a_n = 3n^2 + 7$*. Prove that* $a_n = \Theta(n^2)$*.* *[Hint 165]*

7.19. *Let* a_n *be a sequence and* b_n *a test sequence. Let* $k \in \mathbb{R}$*. Prove that if* $a_n = O(b_n)$*, then* $(ka_n) = O(b_n)$*.* *[Hint 186]*

7.20. *Let* a_n, a'_n *be sequences, and let* b_n *be a test sequence. Suppose that* $a_n = O(b_n)$*, and* $a'_n = O(b_n)$*. Prove that* $(a_n + a'_n) = O(b_n)$*.* *[Hint 103]*

Exercises for Chapter 7.3.

7.21. *An algorithm to print each entry of a balanced binary tree has runtime specified by the recurrence relation* $T_n = 2T_{n/2} + 3$*. Determine what, if anything, the Master Theorem tells us.* *[Hint 146]*

7.22. *Suppose that an algorithm has runtime specified by recurrence relation* $T_n = T_{n/2} + 2^n$*. Determine what, if anything, the Master Theorem tells us.* *[Hint 85]*

7.23. *Suppose that an algorithm has runtime specified by recurrence relation* $T_n = 16T_{n/4} + n$. *Determine what, if anything, the Master Theorem tells us.* *[Hint 179]*

7.24. *Suppose that an algorithm has runtime specified by recurrence relation* $T_n = 4T_{n/2} + n^2$. *Determine what, if anything, the Master Theorem tells us.*

7.25. *Suppose that an algorithm has runtime specified by recurrence relation* $T_n = nT_{n/2} + n^2$. *Determine what, if anything, the Master Theorem tells us.*

7.26. *Suppose that an algorithm has runtime specified by recurrence relation* $T_n = 3T_{n/3} + \sqrt{n}$. *Determine what, if anything, the Master Theorem tells us.*

7.27. *Suppose that an algorithm has runtime specified by recurrence relation* $T_n = 8T_{n/3} + n^2$. *Determine what, if anything, the Master Theorem tells us.* *[Hint 53]*

7.28. *Prove Theorem 7.9. Note: this is Chapter 7.2 material, but there were already a lot of exercises for that chapter.* *[Hint 120]*

7.29. *Prove the unproved parts of Theorem 7.10. (8 parts, some harder than others) Note: this is Chapter 7.2 material, but there were already a lot of exercises for that chapter.* *[Hint 130]*

Exercises for Chapter 8.

Exercises for Chapter 8.1.

8.1. *Give five different names for the set* $\{1, 2, 3, 4\}$*. Be sure to include examples with list, description, and set-builder notation.* [Hint 235]

8.2. *Let* $S = \{1, 2, 3\}$*. Find any set* T*, so that* $S \in T$ *and* $S \subseteq T$ *both hold. Specify* T *in list notation.* [Hint 100]

8.3. *Let* S *be a set. Prove that* $\emptyset \subseteq S$*. Note that, by Theorem 8.14, this means that* $\emptyset \cap S = \emptyset$ *and* $\emptyset \cup S = S$*.* [Hint 214]

8.4. *Let* $S = \{x \in \mathbb{Z} : \exists y \in \mathbb{Z},\ x = 12y\}$*, and* $T = \{x \in \mathbb{Z} : \exists y \in \mathbb{Z},\ x = 3y\}$*. Prove that* $S \subseteq T$*.* [Hint 97]

8.5. *Let* $S = \{x \in \mathbb{Z} : \exists y \in \mathbb{Z},\ x = 12y\}$*, and* $T = \{x \in \mathbb{Z} : \exists y \in \mathbb{Z},\ x = 8y\}$*. Prove that* $S \not\subseteq T$*.* [Hint 81]

8.6. *Let* $S = \{x \in \mathbb{Z} : \exists y \in \mathbb{Z}, x = 24y\}$*, and* $T = \{x \in \mathbb{Z} : \exists y, z \in \mathbb{Z}, x = 8y \wedge x = 3z\}$*. Prove that* $S = T$*.* [Hint 60]

8.7. *Let* $S = \{x \in \mathbb{Z} : \exists y \in \mathbb{Z}, x = 24y\}$*, and* $T = \{x \in \mathbb{Z} : \exists y, z \in \mathbb{Z}, x = 4y \wedge x = 6z\}$*. Prove that* $S \neq T$*.* [Hint 2]

8.8. *Let* R, S, T *be sets. Suppose that* $R \subseteq S$*, and* $S \subseteq T$*. Prove that* $R \subseteq T$*.* [Hint 208]

Exercises for Chapter 8.2.

8.9. *Let* R, S, T *be sets. Draw a Venn diagram representing* $(R \cup S) \cup T$*.* [Hint 63]

8.10. *Let* R, S, T *be sets. Draw a Venn diagram representing* $(R \Delta S) \Delta T$*.* [Hint 27]

8.11. *Let* R, S, T *be sets. Draw a Venn diagram representing* $(R \setminus S) \cup (S \setminus T)$*.* [Hint 239]

8.12. *Let* $S = \{x \in \mathbb{Z} : \exists y \in \mathbb{Z},\ x = 12y\}$, *and* $T = \{x \in \mathbb{Z} : \exists y \in \mathbb{Z},\ x = 8y\}$. *Write* $S \cap T$ *in a nice way, using set-builder notation.* *[Hint 278]*

8.13. *Let* S, T *be sets. Prove that* $S \cap T \subseteq S$. *[Hint 195]*

8.14. *Let* S, T *be sets. Prove that* $S \subseteq S \cup T$. *[Hint 24]*

8.15. *Let* S, T *be sets. Prove that* $S \setminus T \subseteq S$. *[Hint 110]*

8.16. *Let* S *be a set. Prove that* $S \cap S = S$, *and that* $S \cup S = S$. *[Hint 75]*

8.17. *Let* S *be a set. Prove that* $S \Delta \emptyset = S$, *and that* $S \Delta S = \emptyset$. *[Hint 45]*

8.18. *Let* S, T *be sets. Prove that* $S = T$ *if and only if* $S \setminus T = T \setminus S$. *[Hint 15]*

8.19. *Prove the unproved part of Theorem 8.12.* *[Hint 4]*

Exercises for Chapter 8.3.

8.20. *Simplify* $(R \cap S) \cap (S \cap (R \cap S))$ *as much as possible, using the set property theorems and Exercise 8.16.* *[Hint 36]*

8.21. *Prove Theorem 8.13, parts a,b,c,e. If you want a messy challenge, you may do part f (optional).* *[Hint 174]*

8.22. *Prove the unproved parts of Theorem 8.14.* *[Hint 76]*

8.23. *Prove the unproved parts of Theorem 8.15.* *[Hint 257]*

Exercises for Chapter 9.

Exercises for Chapter 9.1.

9.1. *Let* U *be a set, and let* $S \subseteq U$*. Prove that* $S^c \subseteq U$*.* [Hint 39]

9.2. *Prove the unproved parts of Theorem 9.2. (4 parts)* [Hint 190]

9.3. *Let* S, T, U *be sets, with* $S \subseteq U$ *and* $T \subseteq U$*. Prove that* $S \setminus T = S \cap T^c$*.* [Hint 129]

9.4. *Let* S, T, U *be sets, with* $S \subseteq U$ *and* $T \subseteq U$*. Prove that* $S^c \setminus T^c = T \setminus S$*. Hint: use the previous exercise.* [Hint 240]

9.5. *Let* S, T, U *be sets, with* $S \subseteq T \subseteq U$*. Prove that* $T^c \subseteq S^c$*. This is the set theory parallel of Theorem 3.13.*

9.6. *Prove the unproved part of Theorem 9.3.* [Hint 8]

9.7. *Write down twelve different objects: three elements of* $\mathbb{Z}$*, then three subsets of* $\mathbb{Z}$*, then three elements of* $2^{\mathbb{Z}}$*, then three subsets of* $2^{\mathbb{Z}}$*.* [Hint 203]

9.8. *Let* $S = \{x, y, z\}$*. Write* 2^S *in list notation.* [Hint 128]

9.9. *Let* S, T *be sets. Prove that* $S \subseteq T$*, if and only if,* $2^S \subseteq 2^T$*.* [Hint 199]

9.10. *Let* S, T *be sets. Prove that* $2^S \cap 2^T = 2^{S \cap T}$*.* [Hint 54]

9.11. *Find all partitions of set* $S = \{x, y, z\}$*.* [Hint 116]

9.12. *Find a partition of set* $\mathbb{Z}$ *into three parts. Then find a different one.* [Hint 217]

Exercises for Chapter 9.2.

9.13. *Write down twelve different objects: three elements of* $\mathbb{Z}\times\mathbb{Z}$*, then three subsets of* $\mathbb{Z}\times\mathbb{Z}$*, then three elements of* $2^{\mathbb{Z}\times\mathbb{Z}}$*, then three subsets of* $2^{\mathbb{Z}\times\mathbb{Z}}$*.* [Hint 204]

9.14. *Let* $S = \{1, 2, 3\}, T = \{2, 3, 4\}$*. Write down, in list notation,* $(S \times T) \setminus (T \times S)$*.* [Hint 43]

9.15. *Let* A, B, C, D *be sets. Suppose that* $A \subseteq C$*, and that* $B \subseteq D$*. Prove that* $(A \times B) \subseteq (C \times D)$*.* [Hint 61]

9.16. *Prove the unproved parts of Theorem 9.13.* [Hint 91]

9.17. *Let* R, S, T *be sets. Prove that* $(R \cap S) \times T = (R \times T) \cap (S \times T)$*. Similarly the other parts of Theorem 9.13 are true from the other side.*

9.18. *Let* A, B, C, D *be sets. Prove that* $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$*.* [Hint 20]

9.19. *Find sets* A, B, C, D*, for which* $(A\cup B)\times(C\cup D) \neq (A\times C)\cup(B\times D)$*.* [Hint 71]

Exercises for Chapter 9.3.

9.20. *Set* $S = 2\mathbb{Z} = \{2x : x \in \mathbb{Z}\}$*, the set of even integers. Prove that* S *is equicardinal with* $\mathbb{Z}$*.* [Hint 149]

9.21. *Let* A, B *be sets. Prove that* $A \times B$ *is equicardinal with* $B \times A$*.* [Hint 227]

9.22. *Let* A, B, C *be sets. Prove that* $(A \times B) \times C$ *is equicardinal with* $A \times (B \times C)$*.* [Hint 264]

9.23. *Consider the pairing given in Theorem 9.16. Write the first 20 pairs, i.e. the pairs containing naturals* $1, 2, \ldots, 20$*.* [Hint 194]

9.24. *Consider the pairing given in Theorem 9.17. Write the first 20 pairs, i.e. the pairs containing naturals* $1, 2, \ldots, 20$*.* [Hint 159]

9.25. *Prove the unproved part of Corollary 9.19. That is, prove that* $|S| \leq |2^S|$*, by pairing off* S *with a subset of* 2^S*.* [Hint 267]

Exercises for Chapter 10.

For all of the Chapter 10 exercises, let $S^\star = \{1, 2, 3\}$. On this set $S^\star$, define the relations $R_1^\star = \{(1, 2), (2, 3), (3, 2)\}$, and $R_2^\star = \{(1, 1), (2, 2), (3, 3), (2, 3)\}$.

Exercises for Chapter 10.1.

10.1. *Let $S = \{a, b\}$. Find all relations on S.* [Hint 79]

10.2. *Draw digraphs for relations $R_1^\star$ and $R_2^\star$ on $S^\star$.* [Hint 57]

10.3. *Consider relations R_{empty}, R_{full}, and R_{diagonal} on $S^\star$. For each of these three relations, write the relation explicitly in list notation, and draw the corresponding digraph.* [Hint 48]

10.4. *Compute the relations $R_1^\star \cap R_2^\star$ and $R_1^\star \cup R_2^\star$, in list notation, and give their digraphs.* [Hint 25]

10.5. *Determine the relation $R_{\leq} \cap R_{x^2} \cap R_{\lfloor/2\rfloor}$, and write it in list notation.* [Hint 151]

10.6. *Determine the relation $R_{\leq} \cap R_{\lfloor/2\rfloor}$, and write it in set-builder notation.* [Hint 111]

Exercises for Chapter 10.2.

10.7. *Find a set S and a relation R on S, such that R is both reflexive and irreflexive. Hint: S needs to be very special.* [Hint 222]

10.8. *Prove that the relation $R_1^\star$ on $S^\star$ is not symmetric, not antisymmetric, and not trichotomous.* [Hint 74]

10.9. *Let S be a set. Suppose that relation R on S is both symmetric and antisymmetric. Prove that $R \subseteq R_{\text{diagonal}}$.* [Hint 69]

10.10. *Let R be a relation on S. Suppose that R has the property that $\forall x, y \in S, xRy \leftrightarrow yRx$. Prove that $\forall x, y \in S, xRy \rightarrow yRx$.* [Hint 157]

10.11. *Let* R *be a relation on* S. *Suppose that* R *has the property that* $\forall x, y \in S, xRy \to yRx$. *Prove that* $\forall x, y \in S, xRy \leftrightarrow yRx$. *This exercise, together with the previous, proves that the two definitions of symmetry are actually equivalent.* *[Hint 19]*

10.12. *Prove that the two definitions of antisymmetric are logically equivalent.* *[Hint 181]*

10.13. *Prove that the two definitions of trichotomous are logically equivalent.* *[Hint 189]*

10.14. *Let* S *be a set, and* R *a transitive relation on* S. *Use induction to prove that, for all* $n \in \mathbb{N}$ *with* $n \geq 2$, *and any elements* $x_1, x_2, \ldots, x_n \in S$, $(x_1Rx_2 \wedge x_2Rx_3 \wedge \cdots \wedge x_{n-1}Rx_n) \to x_1Rx_n$. *[Hint 33]*

10.15. *Let* S *be a set, and* R *a symmetric relation on* S. *Prove that* R^c *is also symmetric on* S. *[Hint 78]*

Exercises for Chapter 10.3.

10.16. *Compute the relations* $R_1^\star \circ R_2^\star$ *and* $R_2^\star \circ R_1^\star$, *in list notation, and give their digraphs.* *[Hint 38]*

10.17. *Set* $T^\star = \{1, 3\}$. *Compute the relations* $R_1^\star|_{T^\star}$ *and* $R_2^\star|_{T^\star}$, *in list notation, and give their digraphs.* *[Hint 247]*

10.18. *Set* $T = \{1, 2, 3, 4\}$. *Compute the relations* $R_\leq|_T$, $R_{x^2}|_T$, $R_{\lfloor/2\rfloor}|_T$, *in list notation, and give their digraphs.* *[Hint 134]*

10.19. *Let* S *be a set,* $T \subseteq S$, *and* R *a symmetric relation on* S. *Prove that* $R|_T$ *is symmetric.* *[Hint 84]*

10.20. *Let* S *be a set,* $T \subseteq S$, *and* R *a transitive relation on* S. *Prove that* $R|_T$ *is transitive.* *[Hint 124]*

10.21. *Let* S *be a set, and* R *a transitive relation on* S. *Prove that* R^{-1} *is also transitive on* S. *[Hint 161]*

10.22. *Compute the reflexive closure, symmetric closure, and transitive closure, of* $R_1^\star$. *Give them in list notation, and give their digraphs. (3 relations)* *[Hint 104]*

10.23. *Compute the reflexive closure, symmetric closure, and transitive closure, of* $R_2^\star$*. Give them in list notation, and give their digraphs. (3 relations)* *[Hint 202]*

10.24. *Let* S *be a set, and* R *a relation on* S*. Prove that* $(R^{-1})^{-1} = R$*.* *[Hint 22]*

10.25. *Let* S *be a set, with reflexive relation* R*. Prove that* $R \subseteq R \circ R$*.* *[Hint 67]*

10.26. *Let* S *be a set, with transitive relation* R*. Prove that* $R \circ R \subseteq R$*, without using Theorem 10.16.* *[Hint 251]*

10.27. *Let* S *be a set, with relation* R*. Let* R' *be the reflexive closure of* R*. Prove that* R' *is reflexive.* *[Hint 121]*

10.28. *Let* S *be a set, with relation* R*. Let* R' *be the symmetric closure of* R*. Prove that* R' *is symmetric.* *[Hint 49]*

10.29. *Prove the unproved parts of Theorem 10.20.* *[Hint 254]*

Exercises for Chapter 11.

Exercises for Chapter 11.1.

11.1. *Define relation* R *on* $\mathbb{Q}$ *via* $R = \{(a,b) : a - b \in \mathbb{Z}\}$. *Prove that* R *is an equivalence relation.* *[Hint 35]*

11.2. *Define relation* R *on* $\mathbb{Q}$ *via* $R = \{(a,b) : a + b \in \mathbb{Z}\}$. *Prove that* R *is not an equivalence relation.* *[Hint 18]*

For exercises 11.3-11.5, $\mathbb{Z}[x]$ is the set of all polynomials with integer coefficients, in the variable x. Six examples of elements of $\mathbb{Z}[x]$ are:
$5x^2 - 2x + 1, \ -x^3 - 7, \ 100x^{100} + x^{99}, \ x, \ 8, \ 0$
If $p(x) \in \mathbb{Z}[x]$, it is a polynomial, and you can do anything to it that you do with other polynomials. For example, if $p(x) = -x^3 - 7$, we can calculate $p(0) = -0^3 - 7 = -7$, $p(x) + p(x) = -2x^3 - 14$, $p'(x) = -3x^2$.

11.3. *Define relation* R *on* $\mathbb{Z}[x]$ *via* $R = \{(p(x), q(x)) : p(0) - q(0) = 0\}$. *Prove that* R *is an equivalence relation.* *[Hint 187]*

11.4. *Define relation* R *on* $\mathbb{Z}[x]$ *via* $R = \{(p(x), q(x)) : p(0) + q(0) = 0\}$. *Prove that* R *is not an equivalence relation.* *[Hint 82]*

11.5. *Define relation* R *on* $\mathbb{Z}[x]$ *via* $R = \{(p(x), q(x)) : p'(x) = q'(x)\}$. *Here* $'$ *denotes the derivative operator:* $p'(x)$ *is the derivative of* $p(x)$, *and* $q'(x)$ *is the derivative of* $q(x)$. *Prove that* R *is an equivalence relation.* *[Hint 281]*

Exercises for Chapter 11.2.

11.6. *Let* $n \in \mathbb{N}$, *and define relation* R *on* $\mathbb{Z}$ *via* $(a,b) \in R$ *if* a, b *have the same remainder upon dividing by* n *(via the Division Algorithm). Prove that* $(a,b) \in R$, *if and only if,* $a \equiv b \pmod{n}$. *This gives another way of understanding modular equivalence.* *[Hint 283]*

11.7. *Let* $m, n \in \mathbb{N}$. *Suppose that* $x \equiv y \pmod{mn}$. *Prove that* $x \equiv y \pmod{m}$, *and that* $x \equiv y \pmod{n}$. *[Hint 180]*

11.8. *Prove the unproved portions of Theorem 11.4.* *[Hint 93]*

11.9. *Find an integer* $x \in [0, 11)$ *such that* $x \equiv 2^8 \pmod{11}$*. Then find an integer* $y \in [0, 11)$ *such that* $y \equiv x \cdot x \pmod{11}$*. Note that* $y \equiv 2^{16} \pmod{11}$*.* *[Hint 42]*

11.10. *Use the method of Exercise 11.9 to find an integer* $z \in [0, 11)$ *such that* $z \equiv 2^{128} \pmod{11}$*. Do not actually compute* 2^{128}*.* *[Hint 268]*

11.11. *Use the work from Exercise 11.10, and the observation that* $100 = 64 + 32 + 4$*, to find an integer* $z \in [0, 11)$ *such that* $z \equiv 2^{100} \pmod{11}$*. Do not actually compute* 2^{100}*.* *[Hint 147]*

11.12. *Compute* $2^1 \pmod{11}, 2^2 \pmod{11}, 2^3 \pmod{11}, \ldots$*, and find a pattern. Use that pattern to find an integer* $z \in [0, 11)$ *such that* $z \equiv 2^{100} \pmod{11}$*.* *[Hint 70]*

11.13. *Find all integers* $x \in [0, 12)$ *satisfying the modular equation* $5x \equiv 10 \pmod{12}$*.* *[Hint 183]*

11.14. *Find all integers* $x \in [0, 12)$ *satisfying the modular equation* $5x \equiv 11 \pmod{12}$*.* *[Hint 140]*

11.15. *Find all integers* $x \in [0, 12)$ *satisfying the following modular equations (one at a time, not all at once):*

a. $2x \equiv 10 \pmod{12}$

b. $2x \equiv 8 \pmod{12}$

c. $2x \equiv 9 \pmod{12}$

d. $10x \equiv 10 \pmod{12}$

e. $10x \equiv 8 \pmod{12}$

f. $12x \equiv 12 \pmod{12}$

g. $12x \equiv 11 \pmod{12}$

[Hint 263]

11.16. *Use the method given in the proof of the Chinese Remainder Theorem (Theorem 11.8) to solve the linear modular system* $\{x \equiv 5 \pmod 9, x \equiv 1 \pmod{11}\}$. *[Hint 51]*

11.17. *Use the method given in the proof of the Chinese Remainder Theorem (Theorem 11.8) to solve the linear modular system* $\{x \equiv 5 \pmod 9, x \equiv -5 \pmod{11}\}$.

11.18. *Let* $m, n \in \mathbb{N}$, *and set* $d = \gcd(m, n)$. *Let* $b \in \mathbb{Z}$ *with* $d \nmid b$. *Prove that* $mx \equiv b \pmod n$ *has no solutions. [Hint 160]*

11.19. *Let* $m, n \in \mathbb{N}$, *and set* $d = \gcd(m, n)$. *Let* $b \in \mathbb{Z}$ *with* $d|b$. *Prove that* $mx \equiv b \pmod n$ *has exactly* d *solutions in* $[0, n)$. *[Hint 277]*

Exercises for Chapter 11.3.

11.20. *Let* $S = \{0, 1, 2, \ldots, 19\}$. *Draw the graph for equivalence, modulo 3, on* S. *[Hint 123]*

11.21. *Let* $S = \{0, 1, 2, \ldots, 19\}$. *Draw the graph for equivalence, modulo 5, on* S. *[Hint 30]*

11.22. *Consider the equivalence relation from exercise 11.1. Find* $[0.4]$; *give this in set-builder notation, without any direct reference to* R. *[Hint 164]*

11.23. *Consider the equivalence relation from exercise 11.3. Find* $[x^2 + 3x + 1]$; *give this in description notation, without any direct reference to* R. *[Hint 114]*

11.24. *Consider the equivalence relation from exercise 11.5. Find* $[x^2 + 3x + 1]$; *give this in description notation, without any direct reference to* R. *[Hint 119]*

11.25. *Let* $n \in \mathbb{N}$, *and let* $\equiv$ *be the equivalence relation modulo* n, *on* $\mathbb{Z}$. *Let* $a, b \in \mathbb{Z}$. *Define* $[a] + [b] = \{x + y : x \in [a], y \in [b]\}$. *Prove that* $[a] + [b] = [a + b]$. *(equal as sets) [Hint 73]*

11.26. *Let* $n \in \mathbb{N}$, *and let* $\equiv$ *be the equivalence relation modulo* n, *on* $\mathbb{Z}$. *Let* $a, b \in \mathbb{Z}$. *Define* $[a] \cdot [b] = \{t \in \mathbb{Z} : \exists x \in [a], \exists y \in [b], t \equiv xy\}$. *Prove that* $[a] \cdot [b] = [a \cdot b]$. *(equal as sets) [Hint 52]*

Exercises for Chapter 12.

Exercises for Chapter 12.1.

12.1. *Let* $S = \mathbb{Z}$*, and let* R *be the relation of divisibility,* $|$*. Prove that* R *is not a partial order.* [Hint 65]

12.2. *Prove that the three relations in Example 12.3 are partial orders.* [Hint 59]

12.3. *Draw the Hasse diagram for the relation* $|$ *on* $S = \{4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24\}$*.* [Hint 185]

12.4. *Draw the Hasse diagram for the diagonal relation on* $S = \{x, y, z\}$*.* [Hint 266]

12.5. *Consider the relation* $\leq$ *on* $\mathbb{N}$*. Draw the Hasse diagram for the interval poset* $[4, 7]$*.* [Hint 16]

12.6. *Consider the relation* $|$ *on* $\mathbb{N}$*. Draw the Hasse diagram for the interval poset* $[4, 700]$*.* [Hint 101]

Exercises for Chapter 12.2.

12.7. *Consider the relations from Exercises 12.3, 12.4, 12.5, 12.6. Identify any greatest, maximal, least, minimal elements.* [Hint 96]

12.8. *Let* R *be a partial order on set* S*, and* $T \subseteq S$*. Suppose that* a *is greatest in* T*. Prove that* a *is maximal in* T*.* [Hint 47]

12.9. *Let* R *be a partial order on set* S*, and* $T \subseteq S$*. Suppose that* $a, a' \in T$ *are both greatest in* T*. Prove that* $a = a'$*.* [Hint 293]

12.10. *Let* R *be a partial order on set* S*, and* $T \subseteq S$*. Suppose that* $a, a' \in T$ *are both maximal in* T*. Prove that* $a = a' \vee a \parallel a'$*.* [Hint 86]

12.11. *Let* R *be a partial order on set* S*, and* $T \subseteq S$*. Suppose that* $a, a' \in T$*, where* a *is greatest and* a' *is maximal. Prove that* $a = a'$*.* [Hint 225]

12.12. *Consider the partial order* $|$ *on* $\mathbb{N}$*, and set* $T = \{6, 10\}$*. Find three elements* $a, a', a'' \in \mathbb{N}$ *such that* $a|a'$*,* $a \parallel a''$*,* $a' \parallel a''$ *and each of* a, a', a'' *are upper bounds for* T*.* [Hint 288]

12.13. *Consider the partial order* $|$ *on* $\mathbb{N}$*, and set* $T = \{6, 10\}$*. Find three elements* $a, a', a'' \in \mathbb{N}$ *such that* $a \parallel a'$*,* $a \parallel a''$*,* $a' \parallel a''$ *and each of* a, a', a'' *are upper bounds for* T*.* [Hint 230]

12.14. *Let* R *be a partial order on set* S*, and let* $a, b \in S$ *with* aRb*. Prove that the interval poset* $[a, b]$ *has a greatest and a least element.* [Hint 248]

12.15. *Let* R *be a relation on* S*, and let* $a, b \in S$ *with* aRb*. Suppose that* R *is a total order. Prove that the interval poset* $[a, b]$ *is a total order.* [Hint 210]

12.16. *Let* R *be a well-order on* S*. Prove that* R *is a total order.* [Hint 220]

12.17. *Prove that the product order, as given in Definition 12.7.b., is a partial order. For a messy challenge, prove that the lex order in Definition 12.7.a., is a partial order (optional).* [Hint 294]

12.18. *Let* R_1, R_2 *both be the usual order* $\leq$ *on* $\mathbb{N}$*. Let* $T = \{(1, 3), (2, 2), (4, 1)\}$*. Identify any greatest and maximal elements in* T*, in the lex order on* $\mathbb{N} \times \mathbb{N}$*.* [Hint 156]

12.19. *Let* R_1, R_2 *both be the usual order* $\leq$ *on* $\mathbb{N}$*. Let* $T = \{(1, 3), (2, 2), (4, 1)\}$*. Identify any greatest and maximal elements in* T*, in the product order on* $\mathbb{N} \times \mathbb{N}$*.* [Hint 265]

12.20. *Find three partial orders on* $\mathbb{C}$*; one in which* $1 - i < 1 + i$*, one in which* $1 - i > 1 + i$*, and one in which* $1 - i \parallel 1 + i$*.* [Hint 21]

Exercises for Chapter 12.3.

12.21. *Consider the relation* $|$ *on* $S = \{1, 2, 3, 4, 6\}$*. Find all linear extensions of* $|$ *on* S*. (this is really from Chapter 12.2, but that section had too many exercises)* [Hint 286]

12.22. *Consider the relation* $|$ *on* $S = \{1, 2, 3, 5, 6\}$*. Find all linear extensions of* $|$ *on* S*. (this is really from Chapter 12.2, but that section had too many exercises)* [Hint 207]

12.23. *Consider the partial order* $|$ *on* $\{1, 2, 3, \ldots, 10\}$*. Without using Dilworth's Theorem, prove that it has no antichain of size* 6*.* [Hint 212]

12.24. *Set* $T = \{a, b, c, d, e\}$*. Find a maximal clutter from* T*.* [Hint 72]

12.25. *Consider the relations from Exercises 12.3, 12.4, 12.5, 12.6. Find the height and width of each. Be sure to justify.* [Hint 285]

12.26. *Let* R *be a total order on set* S*. Prove that the width of* R *is* 1*, and the height of* R *is* $|S|$*.* [Hint 215]

12.27. *Let* R_1, R_2 *both be the usual order* $\leq$ *on* $S = \{1, 2, 3\}$*. Let* R *be the product order on* $S \times S$*. Find the width and height of* R*. Be sure to prove your answer.* [Hint 126]

Exercises for Chapter 13.

Exercises for Chapter 13.1.

13.1. *Verify the forty properties of relations* $R_1, R_2, \ldots, R_{10}$*, as given in the table below them. Yes, it's a lot!* [Hint 255]

13.2. *Let* $S = [0, 1]$*, an interval in* $\mathbb{R}$*. Find a relation on* S *that is not left-total, not left-definite, not right-total, and not right-definite. Be sure to justify your answer.* [Hint 135]

13.3. *Let* $S = [0, 1]$*, an interval in* $\mathbb{R}$*. Find a relation on* S *that is not left-total and not right-total, but is left-definite and right-definite. Be sure to justify your answer.* [Hint 228]

13.4. *Let* R *be a relation from* S *to* T*; hence,* R^{-1} *is a relation from* T *to* S*. Prove that* R *is left-total if and only if* R^{-1} *is right-total.*

Exercises for Chapter 13.2.

13.5. *Consider the relation* $R : \mathbb{R} \to \mathbb{R}$ *given by* $\{(x, y) : x^2 + y^3 = 1\}$*. Determine whether* R *is a well-defined function.* [Hint 209]

13.6. *Consider the relation* $R : \mathbb{R} \to \mathbb{R}$ *given by* $\{(x, y) : \sin^2 x + \cos^2 x = y\}$*. Determine whether* R *is a well-defined function.* [Hint 108]

13.7. *Consider the relation* $R : \mathbb{R} \to \mathbb{R}$ *given by* $\{(x, y) : y = \tan x\}$*. Determine whether* R *is a well-defined function.* [Hint 131]

13.8. *Consider the relation* $R : \mathbb{Q} \to \mathbb{Q}$ *given by* $\{(x, y) : xy = 1\}$*. Determine whether* R *is a well-defined function.* [Hint 127]

13.9. *Consider the relation* $f : \mathbb{R} \to \mathbb{R}$ *given by* $\{(x, y) : y = \sqrt{x}\}$*. Restrict the domain so that the resulting restricted relation becomes a function.* [Hint 192]

13.10. *Prove Theorem 13.4.* [Hint 5]

13.11. *Prove Theorem 13.9.* *[Hint 139]*

13.12. *Consider the function* $f : \mathbb{Z} \to \mathbb{Z}$ *given by* $f(x) = x + 3$. *Prove that* f *is bijective.* *[Hint 218]*

13.13. *Consider the function* $f : \mathbb{R} \to \mathbb{R}$ *given by* $f(x) = \cos x$. *Prove that* f *is not surjective and not injective.* *[Hint 193]*

13.14. *Consider the function* $f : \mathbb{N} \to \mathbb{N}$ *given by* $f(n) = \frac{n(n+1)}{2}$. *Prove that* f *is injective and not surjective.* *[Hint 243]*

13.15. *Consider the function* $f : \mathbb{R} \to \mathbb{R}$ *given by* $\{(x, y) : y = x^2\}$. *Restrict the domain so that the resulting function becomes injective.*

13.16. *Consider the function* $f : \mathbb{R} \to \mathbb{R}$ *given by* $\{(x, y) : y = x^2\}$. *Restrict the domain and the codomain so that the resulting function becomes bijective.*

13.17. *Prove Theorem 9.16, by proving that the function* $f : \mathbb{N} \to \mathbb{Z}$ *given by* $f : n \mapsto \begin{cases} n/2 & n \text{ is even,} \\ -(n-1)/2 & n \text{ is odd.} \end{cases}$ *is a bijection. Hint: there will be cases, based on even vs. odd.* *[Hint 90]*

Exercises for Chapter 13.3.

13.18. *Let* S, T *be sets, and* $f : S \to T$ *be a function. Prove that* $\mathrm{id}_T \circ f = f$. *[Hint 226]*

13.19. *Let* S, T *be sets, and* $f : S \to T$ *be a function. Prove that* $f \circ \mathrm{id}_S = f$. *[Hint 280]*

13.20. *Prove the unproved parts of Theorem 13.15.* *[Hint 88]*

13.21. *Consider* F_1, F_2, *functions on* $\mathbb{R}$, *given by* $F_1(x) = e^x$, $F_2(x) = x^2$. *Recall that* F_1 *is injective (as proved in Example 13.11). Prove that* F_2 *is not injective, and that* $F_2 \circ F_1$ *is injective. This speaks to the "missing" part of Theorem 13.15.b.* *[Hint 197]*

13.22. *Consider* F_1, F_2*, functions on* $\mathbb{R}$*, given by* $F_1(x) = e^x$*, and* $F_2(x) = \begin{cases} \ln x & x > 0 \\ 5 & x \leq 0 \end{cases}$*. Recall that* F_1 *is not surjective (as proved in Example 13.11). Prove that* F_2 *is surjective, and that* $F_2 \circ F_1$ *is surjective. This speaks to the "missing" part of Theorem 13.15.d.* [Hint 172]

13.23. *Prove the unproved part of Theorem 13.16.* [Hint 233]

13.24. *Let* S, T *be sets, and* $F : S \to T$ *a function. Suppose we had a function* $G : T \to S$ *such that* $G \circ F = \mathrm{id}_S$ *and* $F \circ G = \mathrm{id}_T$*. By the discussion following Theorem 13.16, we know that* F, G *are both bijective. Prove that* $G = F^{-1}$*.* [Hint 117]

13.25. *Consider the function* $f : \mathbb{Z} \to \mathbb{Z}$ *given by* $f(x) = x + 3$*. Use the method of Exercise 13.24 to (re-)prove that* f *is bijective.* [Hint 64]

Hints to Selected Exercises

> Warning: Do not look at any hint until you have made a good-faith effort to solve the exercise on your own. Relying too much on hints will limit your ability to master the material.

1. You will need to start by defining $S = \{m \in \mathbb{Z} : p\}$, where p is a carefully chosen predicate involving m, x. Try various options for p until you find one that works. You need S to have a lower bound to use minimum element induction.

2. You have a choice: Either find some x that is in S and not in T, or find some x that is in T but not in S.

3. You will also need disjunctive syllogism.

4. Part b is simpler than part a, because the propositions that each side work out to, turn out to be the same. There are still two $\subseteq$ relationships to prove.

5. Suppose first that R is right-total. Now prove that the two sets $\mathrm{Im}(R)$ and T are equal ($\subseteq$ and $\supseteq$). Now, suppose that $\mathrm{Im}(R) = T$. Let $y \in T$ be arbitrary. Now prove there is some $x \in S$ with $(x, y) \in R$.

6. Try breaking into two cases: p might be T or F.

7. Write the set down on a grid, then zig-zag through the grid.

8. For one direction: Let $x \in (S \cap T)^c$. Now prove $x \in S^c \cup T^c$. For the other direction: Let $x \in S^c \cup T^c$. Now prove $x \in (S \cap T)^c$.

9. Your truth table will need 16 rows. Yikes!

10. The hypothesis and conclusion don't appear to be related in any way. Perhaps Theorem 3.7?

11. Algebra hint: $(n+1)^3 = n^3+3n^2+3n+1 \geq (2n+1)+3n^2+3n+1 = 3n^2+5n+2$. Now prove $3n^2+5n+2 \geq 2(n+1)+1$.

12. Use induction on n (not x). Algebra hint: Multiply both sides of the inductive hypothesis by $1 + x$.

13. Six of them are neither.

14. Still no need for strong induction. Algebra hint: Work with $F_n^2 - F_{n+1}F_{n-1}$, replacing F_{n+1} by $F_n + F_{n-1}$ and simplifying with the inductive hypothesis.

15. Note that $x \in (S \setminus T) \cap (T \setminus S)$ means that $x \in S \wedge x \notin S$, i.e. this is impossible. Hence $(S \setminus T) \cap (T \setminus S) = \emptyset$.

16. Your diagram should contain four elements and three edges.

17. Many proof structures will work, such as $a \vdash b, b \vdash c, c \vdash d, d \vdash a$.

18. This relation is symmetric, but it is neither reflexive nor transitive. You may choose either of those two properties to disprove.

19. This exercise is more difficult than 10.10. You will need to use your hypothesis twice, with different choices for x, y.

20. There are two directions to prove. For one, let $x \in (A \cap B) \times (C \cap D)$. Hence $x = (u, v)$, an ordered pair, with $u \in A \cap B$ and $v \in C \cap D$. Now prove that $x \in A \times C$ and that $x \in B \times D$, and from there prove that $x \in (A \times C) \cap (B \times D)$. For the other direction, let $x \in (A \times C) \cap (B \times D)$, and prove that $x \in (A \cap B) \times (C \cap D)$.

21. Two of these were given as sample partial orders in the chapter; the third you must construct on your own. Think about choosing a familiar partial order, or its reverse, and combining with another via lex or product order.

22. There are two things to prove. First, suppose that $a, b \in S$ with $(a, b) \in R$. Now prove that $(a, b) \in (R^{-1})^{-1}$. Second, suppose that $a, b \in S$ with $(a, b) \in (R^{-1})^{-1}$. Now prove that $(a, b) \in R$.

23. You can use a truth table for each part.

24. Convert to propositional notation, then use addition.

25. The first relation will contain only one ordered pair; the second will contain six.

26. Algebra hint: add $a + (n+1)d$ to both sides of the inductive hypothesis.

27. Start with an empty Venn diagram with sets R, S, T, then shade $R \Delta S$ in one color, and treat it as one of the sets (together with T) to get your final shading.

28. Algebra hint: Add $2(n+1)+1 = 2n+3$ to both sides of the inductive hypothesis.

29. Start with: Let $x \in \mathbb{N}$ be arbitrary. Hence $x \geq 1$. (etc.)

30. The graph should have five big clumps of vertices, with all vertices connected within each clump.

31. The characteristic polynomial has a double root.

32. You will use modus ponens several times.

33. Your inductive hypothesis is $(x_1 R x_2 \wedge x_2 R x_3 \wedge \cdots \wedge x_{n-1} R x_n) \to x_1 R x_n$. Your desired conclusion is $(x_1 R x_2 \wedge x_2 R x_3 \wedge \cdots \wedge x_{n-1} R x_n \wedge x_n R x_{n+1}) \to x_1 R x_{n+1}$; prove it using a direct proof.

34. Try breaking into two cases: p is T, or r is T.

35. You need to prove that R is reflexive, symmetric, and transitive. For example, to prove that R is transitive, you let $a, b, c \in \mathbb{Q}$ be arbitrary, assume that $(a, b) \in R$ and $(b, c) \in R$, and try to prove that $(a, c) \in R$.

36. You should get $R \cap S$, but remember to justify each step carefully.

37. Use the definition of binomial coefficients. Use the properties of factorial to find a common denominator and add.

38. $(x,z) \in R_1^\star \circ R_2^\star$ with $x = 1, z = 2$, because there is $y = 1 \in S^\star$ with $(x,y) \in R_2^\star$ and $(y,z) \in R_1^\star$

39. Convert to propositional notation, then use simplification.

40. Algebra hint: $10n^2 = n^2 + 2n^2 + 7n^2$, now prove that $2n^2 \geq 2n$ and $7n^2 \geq 1$.

41. You need a vertical line, and a point. Your line should not pass through your point.

42. This and the next two problems show how it is possible for a computer to calculate $a^b \pmod{n}$, when each of a, b, n are hundreds of digits long. It is not possible to calculate a^b (it's too big to store, and too slow to calculate), but $a^b \pmod{n}$ is quick and easy using this algorithm, which is the basis for much of modern cryptography. Multiply, then subtract 11 the right amount of times to leave a result in $[0, 11)$.

43. Note that $(2,2)$ is an element of both $S \times T$ and $T \times S$, but $(1,2)$ is not.

44. The statement is false. Algebra hint: take $x = 12, y = 13$.

45. Two set equalities means four $\subseteq$ relationships to prove. Here's one: Let $x \in S\Delta\emptyset$ be arbitrary. Then $(x \in S \wedge x \notin \emptyset) \vee (x \notin S \wedge x \in \emptyset)$. But $x \in \emptyset$ is false, so $(x \notin S \wedge x \in \emptyset)$ is false, so by disjunctive syllogism $x \in S \wedge x \notin \emptyset$. By simplification, $x \in S$. This proves that $S\Delta\emptyset \subseteq S$.

46. Your proof will have four parts. The first will start by assuming a ($n - 1$ is even), and trying to prove c ($n + 2$ is odd).

47. Assume that a is greatest. Step 1: prove $a \in T$. Step 2: Let $x \in T$ be arbitrary. Apply the hypothesis and addition.

48. R_{empty} will contain no ordered pairs ($R_{empty} = \emptyset$), R_{full} will contain 9 ordered pairs, $R_{diagonal}$ will contain 3 ordered pairs.

49. To prove that $R \cup R^{-1}$ is symmetric, let $(a, b) \in R \cup R^{-1}$. Now there are two cases, $(a, b) \in R$ and $(a, b) \in R^{-1}$. In each case, you need to end with $(b, a) \in R \cup R^{-1}$.

50. The statement is false. You need to find two different integers that satisfy the equation.

51. $9 \times 5 \equiv 1 \pmod{11}$ and $11 \times 5 \equiv 1 \pmod 9$, so take $m = 9, m' = 5, n = 11, n' = 5$.

52. For one direction, let $z \in [a] \cdot [b]$ be arbitrary, and try to prove that $z \in [a \cdot b]$. For the other direction, let $z \in [a \cdot b]$ be arbitrary, and try to prove that $z \in [a] \cdot [b]$.

53. $3^1 = 3$ and $3^2 = 9$.

54. For one direction: Let $x \in 2^S \cap 2^T$. Hence $x \in 2^S \wedge x \in 2^T$. Now prove that $x \in 2^{S \cap T}$. For the other direction: Let $x \in 2^{S \cap T}$. Now prove $x \in 2^S \cap 2^T$.

55. Combine various inequalities to prove $\leq$, then again to prove $\geq$.

56. The statement is true. Begin by assuming $n, n' \in \mathbb{N}$ with $|2n - 1| = 3 = |2n' - 1|$. End with proving $n = n'$.

57. The two digraphs will each have vertices $1, 2, 3$, but different edges.

58. The statement is true. Theorem 3.7 may be helpful.

59. For transitivity in 12.3b, let $A, B, C \in 2^T$ (i.e. $A, B, C \subseteq T$). Suppose that $A \subseteq B$ and $B \subseteq C$. Now, let $x \in A$ be arbitrary. Prove that $x \in C$. This proves that $A \subseteq C$.

60. Prove that $S \subseteq T$, and that $T \subseteq S$. The second one is harder: it helps to use the fact that 3 is prime.

61. Let $x \in A \times B$. Hence $x = (u, v)$, an ordered pair, with $u \in A$ and $v \in B$. Now prove that $x \in C \times D$.

62. $|\sin n| \le 1$. Hence, for $n \in \mathbb{N}$, $|a_n| \le |n^2| + |n| + |1| + |\frac{1}{n}| + |\sin n| \le n^2 + n + 1 + \frac{1}{n} + 1$.

63. Start with an empty Venn diagram with sets R, S, T, then shade $R \cup S$ in one color, and treat it as one of the sets (together with T) to get your final shading.

64. Find a function $g(x)$ so that $g \circ f = id_{\mathbb{Z}} = f \circ g$.

65. R is reflexive and transitive. To prove it is not antisymmetric, you need to find $a, b \in S$ with $(a, b) \in R$ and $(b, a) \in R$ (and $a \neq b$).

66. Two are not well-formed, and three are propositions.

67. To prove $(x, y) \in R \circ R$, you need to find some $z \in S$ with $(x, z) \in R$ and $(z, y) \in R$. Two choices of z stand out.

68. The equation $r^2 - r - 1 = 0$ has two solutions, $\phi = \frac{1+\sqrt{5}}{2}$, and $\phi' = \frac{1-\sqrt{5}}{2}$. It is easier to work with ϕ, ϕ' than with the messy fractions. Note that $\phi + \phi' = 1$, and $\phi - \phi' = \sqrt{5}$.

69. Take arbitrary $(a, b) \in R$. Use the hypotheses you are given to prove that $a = b$; hence $(a, b) \in R_{diagonal}$.

70. Something wonderful will happen after 2^{11}. This pattern is not a coincidence; such a pattern will always exist. To learn more, look up "Fermat's Little Theorem".

71. Examples are plentiful; try $A = \{5\}, B = \{6\}, C = \{7\}, D = \{8\}$.

72. A maximal clutter will be a set of ten subsets of T.

73. To prove set equality, prove $\subseteq$ and $\supseteq$. For one direction, let $z \in [a] + [b]$ be arbitrary. Hence, there exist $x \in [a], y \in [b]$ with $z = x + y$. Using this, try to prove that $z \in [a + b]$. For the other direction, let $z \in [a + b]$ be arbitrary. Hence, $a + b \equiv z \pmod{n}$. Using this, try to find $x \in [a], y \in [b]$ with $z = x + y$; this will prove that $z \in [a] + [b]$.

74. You need three unrelated proofs. To prove that $R_1^\star$ is not symmetric, you must find $a, b \in S^\star$ with $(a,b) \in R_1^\star$ and $(b,a) \notin R_1^\star$. To prove that $R_1^\star$ is not antisymmetric, you must find $a, b \in S^\star$ with $(a,b) \in R_1^\star$ and $(b,a) \in R_1^\star$. To prove that $R_1^\star$ is not trichotomous, you must find $a, b \in S^\star$ with $(a,b) \notin R_1^\star$ and $(b,a) \notin R_1^\star$.

75. Two set equalities means four $\subseteq$ relationships to prove. Here's one: Let $x \in S \cap S$ be arbitrary. Then $x \in S \wedge x \in S$. By simplification, $x \in S$. This proves $S \cap S \subseteq S$.

76. There are two directions to prove. For one, let $x \in S \cup T$ be arbitrary. There will be two cases, and in both you need to get $x \in T$.

77. Using the definitions, you get two inequalities involving ceiling, and two involving floor. You also get the hypothesis that $\lceil x \rceil = \lfloor x \rfloor$. Combine these five tools.

78. Let $a, b \in S$ be arbitrary, and suppose that $(a,b) \in R^c$. Now try to prove that $(b,a) \in R^c$.

79. You should find $2^{|S|^2} = 16$ relations in all.

80. We need columns for p, q, of course, and also for $\neg p, \neg q, (p \wedge \neg q), ((\neg p) \wedge q)$, and $(p \wedge \neg q) \vee ((\neg p) \wedge q)$.

81. Find some specific $x \in S$ such that $x \notin T$.

82. This relation is symmetric, but it is neither reflexive nor transitive. You may choose either of those two properties to disprove.

83. No need for strong induction, just treat this like exercises 6.3-6.6.

84. You will use the definition of symmetric (twice), and the definition of restriction.

85. No k is possible.

86. Argue by contradiction; $\neg(a = a' \vee a \parallel a')$ is equivalent to $a \neq a' \wedge a \nparallel a'$. Now apply the hypotheses to get aRa' and $a'Ra$, which contradicts antisymmetry.

87. You will get lots of practice with De Morgan's Law, and also Double Negation.

88. Suppose that F_1, F_2 are both surjective. Now, $F_2 \circ F_1$ has codomain S_3, so let $z \in S_3$ be arbitrary. Now find $x \in S_1$ so that $(F_2 \circ F_1)(x) = z$. Suppose instead that $F_2 \circ F_1$ is surjective. Let $z \in S_3$ be arbitrary. Now find $y \in S_2$ so that $F_2(y) = z$.

89. The statement is true. Algebra hint: since $x \in \mathbb{N}$, $x \geq 1$, so $2x \geq 2$ and $2x + 1 \geq 3$. Hence $x^2 + 2x + 1 \geq x^2 + 3$.

90. Suppose $f(n) = f(n')$. Case 1: n, n' are both even. Then $n/2 = n'/2$, so $n = n'$. Case 2: n is even, n' is odd. Then $n/2 = -(n' - 1)/2$, a contradiction since the LHS is positive while the RHS is not. There are two more cases, and then you've proved injectivity. Whew!

91. Each of b,c has two directions to prove. For one direction of b, let $x \in R \times (S \cup T)$ be arbitrary. Then $x = (a, b)$, where $a \in R$ and $b \in S \cup T$. Hence $b \in S \vee x \in T$. Two cases: $b \in S$ (in which case $(a, b) \in R \times S$, so by addition $(a, b) \in R \times S \vee (a, b) \in R \times T$), or $b \in T$ (in which case $(a, b) \in R \times T$ so by addition $(a, b) \in R \times S \vee (a, b) \in R \times T$). Both cases allow us to conclude that $x = (a, b) \in (R \times S) \cup (R \times T)$.

92. Still no need for strong induction.

93. For each part, start by assuming that $x_1 \equiv y_1 \pmod{n}$, and that $x_2 \equiv y_2 \pmod{n}$. Then apply the definition of mod (twice), the definition of divides (twice), do some algebra, then apply the definition of divides, and the definition of mod again, to end up with the desired goal.

94. $a - (a - b) = b$

95. First, compute the contrapositive. Then, compute the contrapositive of *that*.

96. Two relations have neither least nor greatest elements.

97. Let $x \in S$. Use properties of S to prove that $x \in T$.

98. Just one line of the truth table is enough, provided it is the right line.

99. There are two things to prove, each by applying a definition.

100. T must contain at least four elements.

101. There are six elements in the interval poset: it contains those $x \in \mathbb{N}$ that satisfy $4|x$ and also $x|700$. For example, it does not contain 8 because $8 \nmid 700$.

102. The characteristic polynomial has two distinct roots, both positive.

103. You are given two M's and two n_0's, and need to find a third M and a third n_0. Use the ones you have to find the new ones.

104. The relations should be sets containing six, four, and six ordered pairs, respectively.

105. Use the quadratic formula on $m^2 + m = (m')^2 + m'$, then eliminate a solution, to prove $m = m'$.

106. You will need to start by defining $S = \{m \in \mathbb{Z} : p\}$, where p is a carefully chosen predicate involving m, a, b. Try various options for p (e.g. $m \leq \frac{a}{b}, m \leq \frac{a}{b} - 1, m < \frac{a}{b}, m < \frac{a}{b} + 1$) until you find one that works. You need S to have an upper bound to use maximum element induction.

107. You need to find one specific $x^\star \in \mathbb{N}$. (turns out there is only one choice that works)

108. For every $x \in \mathbb{R}$, $\sin^2 x + \cos^2 x = 1$.

109. You can mimic the proof of part a, or (more cleverly) *use* the proof of part a, together with exercise 5.20.

110. Let $x \in S \setminus T$ be arbitrary. Now try to prove that $x \in S$.

111. Your relation must contain infinitely many ordered pairs, of two types. Your solution should look like: $R = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : p \vee q\}$, for appropriately chosen predicates p and q (each depending on x, y).

112. Start by assuming a, b are odd, and $a + b$ is not even. End by finding that $a + b$ *is* even, which is a contradiction.

113. One will have no order, one will have zero-th order, three will have second order, one will have third order.

114. For $p(x) = x^2 + 3x + 1$, we have $p(0) = 0^2 + 3 \cdot 0 + 1 = 1$. Hence, if $(p(x), q(x)) \in R$, we should have $q(0) = 1$ as well. Your set should contain infinitely many polynomials, including $x^2 + 3x + 1$.

115. You can mimic the proof of part a, or (more cleverly) *use* the proof of part a, together with exercise 5.20.

116. There are five partitions for you to find.

117. G and F^{-1} both have domain T. Let $x \in T$ be arbitrary, and set $y_1 = G(x)$, $y_2 = F^{-1}(x)$. Now prove that $y_1 = y_2$.

118. Now you need strong induction. Algebra hint: $1.5^2 = 2.25 < 2.5 = 1.5^1 + 1.5^0$.

119. For $p(x) = x^2 + 3x + 1$, we have $p'(x) = 2x + 3$. Note that $\int p'(x)dx = \int 2x + 3dx = x^2 + 3x + C$. Your set should contain infinitely many polynomials, including $x^2 + 3x + 1$.

120. You are given two M's and two n_0's, and need to find a third M and a third n_0. Use the ones you have to find the new ones.

121. Start with: let $x \in S$ be arbitrary. End with: hence $(x, x) \in R'$.

122. All the parts are short, and matters of perspective. That is, when p, q are any propositions, one might take $p =$'p', $q =$'q'; or $p =$'p', $q =$'p'; or $p =$'q', $q =$'p', etc.

123. The graph should have three big clumps of vertices, with all vertices connected within each clump.

124. Let $a, b, c \in T$, and suppose that $(a, b) \in R|_T$ and $(b, c) \in R|_T$. Now prove that $(a, c) \in R|_T$.

125. The characteristic polynomial has two distinct roots, one positive and one negative.

126. The height and width add up to 8.

127. Think about $x = 0$.

128. Your set should have $2^3 = 8$ elements.

129. Prove $\subseteq$ and $\supseteq$ separately.

130. Algebra hint 1: take logs of $n^\nu \leq M2^n$, getting $\nu \ln n \leq \ln M + n \ln 2$, which rearranges to $n \ln 2 - \nu \ln n \geq -\ln M$. Now $\lim_{n\to\infty} n \ln 2 - \nu \ln n = \lim_{n\to\infty} n(\ln 2 - \nu \frac{\ln n}{n}) = \infty$ (use L'Hopital's rule). Algebra hint 2: part (d) is similar to part (e), except n is replaced by 2 in the base of the exponential.

131. The answer is no; now prove it.

132. You will have two cases, either by Corollary 1.8 or by the Division Algorithm Theorem.

133. First compute the converse. Then, compute the inverse of *that*. Lastly, compute the contrapositive of what you just found.

134. The relations should be sets containing ten, two, and six ordered pairs, respectively.

135. Try $(x - \frac{1}{2})^2 + (y - \frac{1}{2})^2 = \frac{1}{9}$.

136. A fully simplified expression will have $\neg(p \to q)$ replaced by $p \wedge (\neg q)$. (for suitable propositions p, q).

137. Many solutions are possible. One, for $a, b, c \in \mathbb{Z}$, is $\forall a, a + b = c$. Find another.

138. Start with: Let $x \in \mathbb{N}$ be arbitrary. Now set y based on a side calculation (it can depend on x), to allow the algebra to work out. With all this, prove $y \geq x$.

139. Suppose first that R^{-1} is a function. Prove that R is right-total, and then prove that R is left-definite. Next, suppose that R is both right-total and left-definite. Prove that R^{-1} is left-total and right-definite (i.e. a function).

140. Modulo 12, we have $11 \equiv 23 \equiv 35 \equiv 47 \equiv \cdots$. So, we can replace 11 with a more convenient number.

141. It's easier to get a piecewise-defined formula, for odd and even n separately.

142. The definition of floor and ceiling give you four inequalities, but you only need two of them.

143. q will not be -33, because we need $0 \leq r < 3$.

144. A fully simplified expression will have $p \leftrightarrow q$ replaced by $(p \wedge \neg q) \vee (q \wedge \neg p)$. (for suitable propositions p, q).

145. A first-order recurrence must be of the form $a_n = ca_{n-1}$, for some constant $c \in \mathbb{R}$. It will have characteristic polynomial $r - c$.

146. c_n is small.

147. $2^{100} = 2^{64+32+4} = 2^{64}2^{32}2^4$.

148. A fully simplified expression will have $(x \leq y) \wedge (y < z)$ replaced by $(x > y) \vee (y \geq z)$.

149. You need a pairing of the elements of S with the elements of $\mathbb{Z}$: there is one that is screaming out as the natural one.

150. One way: use Theorem 5.20.

151. Your relation should contain just two ordered pairs.

152. You will also need additional semantic theorems.

153. You need to find one specific $x^\star \in \mathbb{N}$. (pick one out of the two that work)

154. You need to prove $\neg(a_n = O(n^2))$, i.e. $\forall n_0 \in \mathbb{N}, \forall M \in \mathbb{R}, \exists n \geq n_0,\ |a_n| > M|n^2|$. Hence, your proof should begin with: let $n_0 \in \mathbb{N}, M \in \mathbb{R}$ be arbitrary. Now find an n (via side calculation) that simultaneously satisfies $n \geq n_0$ and $|n^{2.1}| > M|n^2|$. (your n will, of necessity, depend on both n_0 and M).

155. Three cases, $r = 0, 1, 2$. With $r = 1$, we have $n = 3q+1$, so $n-1 = 3q$.

156. There is a greatest element.

157. Try Theorem 2.17.

158. Multiple uses of conditional interpretation, among other rules.

159. To get you started: $1 = 2^0(2 \times 0 + 1)$, so $1 \leftrightarrow (0, 0)$. $2 = 2^1(2 \times 0 + 1)$, so $2 \leftrightarrow (1, 0)$. $3 = 2^0(2 \times 1 + 1)$, so $3 \leftrightarrow (0, 1)$. $4 = 2^2(2 \times 0 + 1)$, so $4 \leftrightarrow (2, 0)$.

160. Argue by contradiction. Suppose $x \in \mathbb{Z}$ satisfied the modular equation. Use the definition of modular equivalence, the definition of divides, a bit of algebra, and find an equation where d divides one side but not the other.

161. Let $a, b, c \in S$, and suppose that $(a, b) \in R^{-1}$ and $(b, c) \in R^{-1}$. Now prove that $(a, c) \in R^{-1}$.

162. It is not enough to provide a truth table. We must also justify certain rows being removed, and interpret what remains.

163. You need to prove $\neg(a_n = O(2.1^n))$, i.e. $\forall n_0 \in \mathbb{N}, \forall M \in \mathbb{R}, \exists n \geq n_0,\ |2.1^n| > M|2^n|$.

164. Your set should be infinite, and contain $0.4, 1.4, 2.4, -0.6$, and -1.6.

165. There are two things to prove, using two different M.

166. You will need to start by defining $S = \{m \in \mathbb{Z} : p\}$, where p is a carefully chosen predicate involving m, a, b. Try various options for p (e.g. $m \geq \frac{a}{b}, m \geq \frac{a}{b} - 1, m > \frac{a}{b}, m > \frac{a}{b} + 1$) until you find one that works. You need S to have a lower bound to use minimum element induction.

167. Three cases: $x < -1$, $-1 \leq x \leq 1$, $x > 1$. With $x < -1$, $|x-1| = -x+1$ and $|x+1| = -x-1$. For a quick refresher on absolute value, see the Appendix (p. 221).

168. Algebra hint: Add $(n+1)^2$ to both sides of the inductive hypothesis.

169. Use the first-order algorithm you worked out in exercise 7.11.

170. Try using Thm 5.19.

171. Start with the base case, $n = 0$: prove that $2^n > n$. Then, let $n \in \mathbb{N}_0$ be arbitrary, assume that $2^n > n$, and try to prove that $2^{n+1} > n+1$.

172. F_2 is surjective because $\ln x$ is already surjective, so for arbitrary $y \in \mathbb{R}$ there is some $x > 0$ with $F_2(x) = y$. However, F_1 is not surjective, because $e^x > 0$ for every x. Hence you can find $z \in \mathbb{R}$ so that there is no $x \in \mathbb{R}$ with $(F_2 \circ F_1)(x) = z$.

173. You can mimic the proof of part a, or (more cleverly) *use* the proof of part a, together with exercise 5.20.

174. Commutativity is simpler to prove than associativity. In all cases, you have two (very similar) directions to prove.

175. Algebra hint: Add $\frac{1}{(n+1)(n+2)}$ to both sides of the inductive hypothesis.

176. Algebra hint: For $n \in \mathbb{N}$, $\frac{1}{n} \leq 1$ and $\frac{1}{n+1} \leq 1$.

177. $10! = 10 \cdot 9!$.

178. $\pi \approx 3.14$ and $\pi^2 \approx 9.9$.

179. c_n is small.

180. First use the definition of $\equiv$, then use the definition of $|$.

181. Try Theorem 2.15.

182. Mimic the proof that $\sqrt{2}$ is irrational, and use the fact that 3 is prime. Note that even and odd have nothing to do with this problem.

183. There's only one.

184. There are three hypotheses, all of which must be used to get the conclusion. You will need to apply the definition four times.

185. Your diagram should have five numbers on the bottom row, four on the second row, and two on the top row.

186. You are given n_0, M from your hypothesis ($a_n = O(b_n)$). Using these, you need to find n_0', M', such that $\forall n \geq n_0', |ka_n| \leq M'|b_n|$.

187. To prove that R is reflexive, let $p(x) \in \mathbb{Z}[x]$ be arbitrary. Now prove that $(p(x), p(x)) \in R$. To prove that R is symmetric, let $p(x), q(x) \in \mathbb{Z}[x]$ be arbitrary, and assume that $(p(x), q(x)) \in R$. Now prove that $(q(x), p(x)) \in R$. To prove that R is transitive, let $p(x), q(x), f(x) \in \mathbb{Z}[x]$ be arbitrary, and assume that $(p(x), q(x)) \in R$ and $(q(x), f(x)) \in R$. Now prove that $(p(x), f(x)) \in R$.

188. You can either repeat the proof of part a, or (more cleverly) *use* the proof of part a, with a change of perspective.

189. Try Theorem 2.15.

190. For part b, there are two things to prove. For the first direction, let $x \in S \cup S^c$ be arbitrary. Hence $x \in S \vee x \in S^c$. We have two cases: $x \in S$ (and since $S \subseteq U$, $x \in U$), or $x \in S^c$ (so $x \in U \setminus S$ and thus $x \in U \wedge x \notin S$, so by simplification $x \in U$). In both cases $x \in U$. For the second direction, let $x \in U$ be arbitrary. We have two cases:

$x \in S$ (so $x \in S \vee x \in S^c$ by addition), or $x \notin S$ (so $x \in U \wedge x \notin S$ by conjunction, hence $x \in S^c$, hence $x \in S \vee x \in S^c$ by addition). In both cases, $x \in S \vee x \in S^c$, so $x \in S \cup S^c$.

191. You will need to start by defining $S = \{m \in \mathbb{Z} : p\}$, where p is a carefully chosen predicate involving m, x. Try various options for p (e.g. $m \geq x, m \geq x - 1, m > x, m > x + 1$) until you find one that works. You need S to have a lower bound to use minimum element induction.

192. We can only take square roots of nonnegative real numbers.

193. To prove that f is not surjective, find some $y \in \mathbb{R}$ and use the properties of $f(x)$ to prove that $\forall x \in \mathbb{R},\ f(x) \neq y$.

194. To get you started: $1 \leftrightarrow 0$ (since $0 = -(1-1)/2$), $2 \leftrightarrow 1$ (since $1 = 2/2$), $3 \leftrightarrow -1$ (since $-1 = -(3-1)/2$), $4 \leftrightarrow 2$ (since $2 = 4/2$).

195. Convert to propositional notation, then use simplification.

196. One way: write $\lfloor x + \frac{1}{2} \rfloor = \lfloor \lfloor x \rfloor + x - \lfloor x \rfloor + \frac{1}{2} \rfloor$, apply Corollary 1.8, and consider cases.

197. To prove that F_2 is not injective, you need to find x_1, x_2 so that $F_2(x_1) = F_2(x_2)$. However, only one of x_1, x_2 will be in the image of F_1. To prove that $F_2 \circ F_1$ is injective, let $x_1, x_2 \in \mathbb{R}$ be arbitrary. Suppose that $F_2 \circ F_1(x_1) = F_2 \circ F_1(x_2)$, and try to prove that $x_1 = x_2$.

198. The statement is true. You need to find the one specific $x^\star \in \mathbb{N}$ that works.

199. For one direction: Assume $S \subseteq T$. Let $x \in 2^S$. Now prove that $x \in 2^T$. For the other direction: Assume $2^S \subseteq 2^T$. Let $x \in S$. Now prove that $x \in T$.

200. Start by assuming that a is irrational and $a + 2$ is not irrational, i.e. rational. End by finding that a is not irrational, which is a contradiction.

201. Prove the two inequalities separately.

202. The relations should each be a set containing four or five ordered pairs.

203. You will have three numbers, six sets of numbers, and three sets of sets of numbers.

204. You will have three ordered pairs of numbers, six sets of ordered pairs of numbers, and three sets of sets of ordered pairs of numbers.

205. Your nemesis gives you some x, y, and you need to use your knowledge of these to find a z to make the inequality true.

206. To prove that a curve is even+, you need to find a vertical line with a special property. To prove that a curve is not even+, you need to prove that no such vertical line can exist.

207. There are eight linear extensions.

208. Assume that $R \subseteq S$ and $S \subseteq T$. Let $x \in R$ be arbitrary. Now prove that $x \in T$.

209. The answer is yes; now prove it.

210. Let $x, y \in [a, b]$. Prove that $xR'y \vee yR'x$ in the interval poset relation $R' = R|_{[a,b]}$. You will need Definition 10.11.

211. First compute the converse. Then, compute the converse of *that*.

212. There will be a bunch of cases (Dilworth's theorem is very valuable!)

213. The answer is yes, now explain which and why.

214. $(x \in \emptyset) \to (x \in S)$ is vacuously true.

215. To prove the width is 1, prove that no antichain of size 2 can exist (there is always an antichain of size 1). To prove the height is $|S|$, find a chain of size $|S|$ (no chain can be larger).

216. You will need to pick M to be larger than $1,000,000$.

217. One way: use the division algorithm. Another way: primes/composites/other. Another way: positive/negative/zero.

218. To prove that f is injective, suppose that $f(x_1) = f(x_2)$, and prove that $x_1 = x_2$. To prove that f is surjective, let $y \in \mathbb{Z}$ be arbitrary and find some $x \in \mathbb{Z}$ with $f(x) = y$.

219. The statement is true. Start with: Let $x \in \mathbb{N}$ be arbitrary.

220. Let $a, b \in S$. Set $T = \{a, b\}$, and apply the well-order property.

221. The hypothesis and conclusion don't appear to be related in any way. Perhaps Theorem 3.7?

222. There is only one S that can work.

223. You can mimic the proof of part a, or (more cleverly) *use* the proof of part a, together with exercise 5.20.

224. There are two hypotheses, both of which must be used to get the conclusion.

225. Similar to 12.9. Once $a'Ra$, we know that a, a' are not parallel.

226. They both have domain S. Now determine how each of them acts on arbitrary $x \in S$.

227. You need a pairing of the elements of $A \times B$ (which are ordered pairs (u, v) with $u \in A$ and $v \in B$), with the elements of $B \times A$. There is one very natural pairing.

228. Try $\{(x, y) : y = 7x, -0.1 \leq x \leq 0.1\}$.

229. It would be a good idea to look up these terms (in some outside source, not this text) even if you are fairly sure you know their definitions already.

230. Try $210, 300, 330$.

231. Both parts are matters of perspective. That is, when p, q are any propositions, one might take $p =$'p', $q =$'q'; or $p =$'p', $q =$'p'; or $p =$'q', $q =$'p', etc.

232. The statement is true. Algebra hint: take $z = \frac{x+y}{2}$. Why won't this same strategy work for the preceding problem?

233. Mimic the proof of part (a); show that the two functions have the same domain and act the same way on each element of that domain.

234. You get two tools: $a \leq b$ and $b < c$. Using these, you need to prove two things: $a + d \leq b + d$, and also $b + d < c + d$.

235. Another name in list notation is $\{1, 2, 2, 4, 3\}$.

236. First step: $(n+2)! = (n+2) \cdot (n+1)!$.

237. Make the truth table, and see if the corresponding columns agree or not.

238. You will also need modus ponens.

239. Start with an empty Venn diagram with sets R, S, T, then shade $R \setminus S$ in one color and $S \setminus T$ in another color. Treat these two shaded regions as your two sets in getting your final shading.

240. Prove that each side equals $S^c \cap T$.

241. Use a contrapositive proof.

242. Your nemesis gives you some x, and you need to use your knowledge of this to find y, z to make the inequality true.

243. Algebra hint: From $x^2 + x = y^2 + y$, we complete the square to get $(x + \frac{1}{2})^2 - \frac{1}{4} = (y + \frac{1}{2})^2 - \frac{1}{4}$, and hence $|x + \frac{1}{2}| = |y + \frac{1}{2}|$.

244. Try using Corollary 1.8.

245. The answer is yes, now explain which and why.

246. First state and simplify the negation, then prove that.

247. $(x, y) \in R_2^\star|_{T^\star}$ with $x = 1, y = 1$, because $(x, y) \in R_2^\star$ and also $x, y \in T^\star$.

248. a will be least, and b will be greatest: use the definition of the interval poset.

249. Algebra hint: add ar^{n+1} to both sides of the inductive hypothesis.

250. Try breaking into two cases: q is F, or s is F.

251. You can't use Theorem 10.16, but you can read its proof to give you a strategy.

252. The statement is false. You need to find one specific $x^\star \in \mathbb{N}$. (pick one out of the three that work)

253. Existence is by definition of even. Uniqueness is slightly harder.

254. Each set is automatically a subset of its reflexive closure, and also of its symmetric closure. This is the easy direction. For the hard direction, take arbitrary (a, b) in the reflexive/symmetric closure of R, and try to prove $(a, b) \in R$. [prove the two parts separately, don't try to do reflexive and symmetric at the same time.]

255. For R_1: To prove left-total, let $x \in S$ be arbitrary. Set $y = +\sqrt{1 - x^2}$ (the positive square root). We have $y \geq 0$ and $x^2 + y^2 = 1$, so $(x, y) \in R_1$. To prove right-definite, let $x \in S$, $y_1, y_2 \in T$, with $(x, y_1) \in R_1$ and $(x, y_2) \in R_1$. Then $x^2 + y_1^2 = 1 = x^2 + y_2^2$, so $y_1^2 = y_2^2$ so $|y_1| = |y_2|$. Since $y_1, y_2 \geq 0$, we must have $y_1 = y_2$. To prove it is not right-total, $y = -0.5 \in T$, but there is no $x \in S$ with $(x, -0.5) \in R_1$ (since $-0.5 \geq 0$ is false). To prove it is not left-definite, note that $(-1, 0)$ and $(1, 0)$ are both in R_1, but $-1 \neq 1$. 4 down, 36 to go.

256. Start with the base case, $n = 1$: prove that $3^n > 2^n$. Then, let $n \in \mathbb{N}$ be arbitrary, assume that $3^n > 2^n$, and try to prove that $3^{n+1} > 2^{n+1}$.

257. You can mimic the proof of 8.15, or you can rely on distributivity of propositional calculus.

258. One possible proof structure is $a \vdash b, b \vdash c, c \vdash d, d \vdash a$. Now find four more.

259. Use a direct proof.

260. Algebra hint: Add $(-1)^{n+1}(n+1)^2$ to both sides of the inductive hypothesis.

261. You will have two cases, either by Corollary 1.8 or by the Division Algorithm Theorem.

262. Four cases: $x < -1$, $-1 \leq x \leq 0$, $0 \leq x \leq 1$, $x > 1$.

263. (a) There are two. (c) Prove that $2x - 9$ is not even, for every integer x.

264. You need a pairing of the elements of $(A \times B) \times C$ (which are ordered pairs $((u, v), w)$, with $(u, v) \in A \times B$ and $w \in C$), with the elements of $A \times (B \times C)$. There is one very natural pairing.

265. There is no greatest element.

266. Your diagram should have no edges.

267. Try pairing each element $x \in S$ with the set containing just that element, $\{x\}$.

268. $y \cdot y \equiv 2^{32} \pmod{11}$, then keep going, reducing modulo 11 at each step.

269. A fully simplified expression will have $=$ replaced by $\neq$.

270. Still no need for strong induction.

271. You will need to start by defining $S = \{m \in \mathbb{Z} : p\}$, where p is a carefully chosen predicate involving m, a, b. Try various options for p until you find one that works. You need S to have a lower bound to use minimum element induction.

272. With $n = 2$, we have $a_2 = 5a_1 - 6a_0 = 5 \times 1 - 6 \times 1 = -1$. With $n = 3$, we have $a_3 = 5a_2 - 6a_1 = 5 \times (-1) - 6 \times 1 = -11$.

273. For the contradiction, you can use your knowledge that $\frac{1}{2}$ is not an integer.

274. We need to consider p, q, of course, and also $p \wedge q$, $\neg p$, and $(p \wedge q) \wedge (\neg p)$.

275. For (c), prove that $ac - bc \in \mathbb{N}_0$.

276. Try breaking into two cases: q might be T or F.

277. Reduce the problem to a modular equation, mod $\frac{n}{d}$, with a unique solution (modulo $\frac{n}{d}$). Now write out all the integer solutions, and find how many are in the interval $[0, n)$.

278. $\text{lcm}(8, 12) = 24$.

279. Your proofs should be similar to the proofs of the other parts.

280. Similar to exercise 13.18.

281. You can solve this problem without knowing anything about derivatives. Alternatively, you can write $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x^1 + a_0$, with some coefficients a_i, and then $p'(x) = n a_n x^{n-1} + (n-1) a_{n-1} x^{n-2} + \cdots + 1 a_1 x^0$.

282. Algebra hint: Multiply both sides of the inductive hypothesis by $\frac{(2n+2)(2n+1)}{(n+1)(n+1)}$.

283. For any $a, b \in \mathbb{Z}$, we may apply the division algorithm twice to get $a = nq_1 + r_1$ and $b = nq_2 + r_2$. For one direction, we assume that $(a, b) \in R$, i.e. $r_1 = r_2$, and we try to prove that $a \equiv b \pmod{n}$. For the other direction, we assume that $a \equiv b \pmod{n}$, then we try to prove $r_1 = r_2$.

284. Break the proof into two cases: either p is T, or p is F. In each case, prove that p, q have the same truth value. The second case is harder. Alternatively, start with a two-column truth table and eliminate rows as in the other proofs of this section.

285. Remember that to justify the width requires both a large antichain and the use of Dilworth's theorem. The height and width of $[4, 700]$ add up to 6.

286. There are five linear extensions.

287. The statement is false. Start with: Let $x \in \mathbb{N}$ be arbitrary. Two cases $x \leq 2$ or $x \geq 3$. If $x \leq 2$, then $3x \leq 6$ so $3x - 8 \leq -2 < 0$ so $|3x - 8| = -(3x - 8)$. (etc.)

288. Try $210, 330, 3300$.

289. For 3.14, use a direct proof. For 3.15, use a contrapositive proof together with Cor. 1.8.

290. Still no need for strong induction.

291. Algebra hint: Multiply both sides of the inductive hypothesis by $n+1$.

292. The characteristic polynomial has two distinct roots, one positive and one negative.

293. Use the hypotheses to prove that aRa' and $a'Ra$, then apply antisymmetry.

294. You must prove reflexive, antisymmetric, and transitive. To prove antisymmetry, suppose that $(a, b) \leq (c, d)$ and $(c, d) \leq (a, b)$. Use the fact that R_1, R_2 are partial orders, to conclude that $a = c$ and $b = d$, hence $(a, b) = (c, d)$.

295. For $y = x^2$, think about how the curve does not dip below the x-axis, and yet $\lim_{x \to \infty} x^2 = +\infty$.

296. Start with: Let $x \in \mathbb{N}$ be arbitrary. Two cases: $x = 1$ or $x \geq 2$. If $x = 1$, then $3x - 5 = -2 < 0$ so $|3x - 5| = -(3x - 5)$. (etc.)

297. Your truth table will need 16 rows. Yikes!