Name: RED ID:

Spring 2023 Math 245 Final Exam

Please read the following directions:

Please write legibly, with plenty of white space. Please **print** your name and REDID in the designated spaces above. Please fit your answers into the designated areas; material outside the designated areas (such as on this cover page) will not be graded. To get credit, you must also show adequate work to justify your answers. If unsure, show the work. The first four questions are worth 6-12 points, and the remaining sixteen questions are worth 10-20 points. The maximum possible score is $4 \times 12 + 16 \times 20 = 368$, and you get two bonus points for free, for a total of 370 points. The use of notes, books, calculators, or other materials on this exam is strictly prohibited, except you may bring one 8.5”x11” page (both sides) with your handwritten notes. If you need scratch paper, you may use any blank space on your note sheet and on this front page. This exam will begin at 10:30 and will end at 12:30; pace yourself accordingly. Good luck!

Special exam instructions for SSW-1500:

1. Please stow all bags/backpacks/boards at the front of the room. PLEASE DO NOT BLOCK THE ENTRANCE AREA. All contraband, except phones, must be stowed in your bag. All phones and smartwatches must be silent, non-vibrating, and either in your pocket or stowed in your bag.
2. Please remain quiet to ensure a good test environment for others.
3. Please keep your exam on your desk; do not lift it up for a better look.
4. If you have a question or need the restroom, please come to the front. Bring your exam.
5. If you are done and want to submit your exam and leave, please wait until one of the designated exit times, listed below. Please do **NOT** leave at any other time. If you are sure you are done, just sit and wait until the next exit time, with this cover sheet visible.

Designated exam exit times:

10:50 “See you next semester”
11:10 “I wish I had studied more”
11:30 “One extra hour of drinking – worth it”
11:50 “Maybe this will be good enough”
12:10 “There is nothing more in my brain, let me out of here”
12:30 “I need every second I can get”

REMINDER: Use complete sentences.

Problem 1. Carefully state the following definitions:

a. factorial

b. floor

Problem 2. Carefully state the following definitions:

a. tautology

b. xor

Problem 3. Carefully state the following definitions:

a. converse

b. big O

Problem 4. Carefully state the following definitions:

a. symmetric difference

b. right total

Problem 5. Let $n \in \mathbb{N}$. Prove that if n is composite, then $2n$ is composite.

Problem 6. Let $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$. Prove the following:

$$\forall \varepsilon \in \mathbb{R}^+ \; \exists \delta \in \mathbb{R}^+ \; \forall x \in \mathbb{R}, \; |x - 3| < \delta \to |2x - 6| < \varepsilon$$

Problem 7. Use induction to prove that for all $n \in \mathbb{N}$ with $n \geq 3$, we have $n^2 \geq 2n + 1$.

Problem 8. Prove or disprove: $\forall x \in \mathbb{R}, \; \lfloor -10x \rfloor = -10\lceil x \rceil$.

Problem 9. Find a recurrence relation for sequence T_n so that the Master Theorem would tell us $T_n = \Theta(n^{3/2} \log n)$.

Problem 10. Prove or disprove: For all sets A, B, C, if $A \cup B \subseteq B \cap C$, then $A \subseteq C$.

Problem 11. Prove that for any set S, we must have $|S| \leq |2^S|$.
NOTE: This is not Cantor's Theorem – it is used to prove a corollary.

Problem 12. Find a relation on $S = \{a, b, c, d\}$ that is trichotomous and antisymmetric. Give your relation both as a set and as a digraph.

Problem 13. Find all integers $x \in [0, 36)$ satisfying $15x \equiv 9 \pmod{36}$.

Problem 14. Let R, S be sets, and let$F_1 : R \to S$ and $F_2 : S \to R$ be functions. Prove that $F_2 \circ F_1$ is a well-defined function.

Problem 15. Consider the function $f : \mathbb{N} \to \mathbb{Z}$ given by $f(n) = \begin{cases} n/2 & n \text{ even} \\ -(n-1)/2 & n \text{ odd} \end{cases}$. Prove or disprove that f is surjective.

Problem 16. Consider the function $f : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{N}$ given by $f((x, y)) = 3^x(3y + 1)$. Prove or disprove that f is injective.

For all the remaining problems, 17-20, we let S be the set of all equivalence relations on $\mathbb{Z}$, i.e. $S = \{R \subseteq \mathbb{Z} \times \mathbb{Z} : R \text{ is an equivalence relation}\}$. We consider the subset partial order $\subseteq$ on S, i.e. $\subseteq = \{(R_1, R_2) \in S \times S : R_1 \subseteq R_2\}$.

Problem 17. Prove or disprove that $\equiv_4 \subseteq \equiv_2$.

Problem 18. Find $R_1, R_2 \in S$ satisfying $R_1 \| R_2$.

Problem 19. Determine, with proof, whether $\subseteq$ has a maximum and/or a minimum on S.

Problem 20. Determine, with proof, the height of this poset.