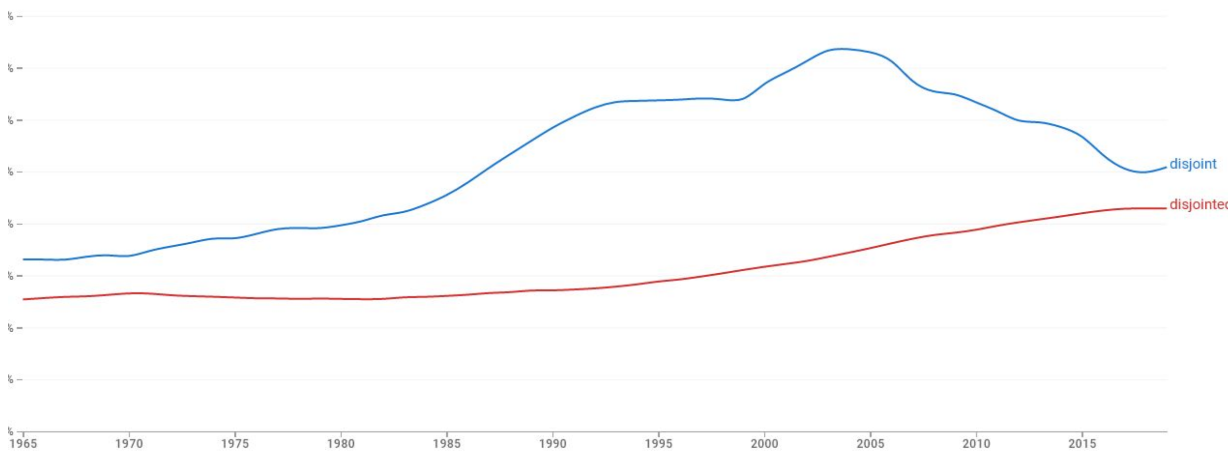
MATH 245 S23, Exam 3 Solutions

1. Carefully define the following terms: = (for sets), union.
   Two sets are equal if they contain exactly the same elements. Given two sets S, T, their union (denoted $S \cup T$) is the set given by $\{x : x \in S \vee x \in T\}$.

2. Carefully define the following terms: disjoint, trichotomous
   Two sets are disjoint if their intersection equals the empty set. Let R be a relation on set S. We say that R is trichotomous if (either option is correct):
   OPTION 1: $\forall x, y \in S,\ (x = y) \vee (xRy) \vee (yRx)$.
   OPTION 2: $\forall x, y \in S,\ (x \not R y \wedge y \not R x) \rightarrow (x = y)$.

   NOTE: For some reason, many of you wrote "disjointed" even though the word "disjoint" was written out for you right there. I did not take points off, but I am baffled. "disjoint" is a common word, why confuse it with a less common one? Here's what Google Ngram has to say:



3. Let S, U be sets with $S \subseteq U$. Prove that $S \subseteq (S^c)^c$
   NOTE: This is part of Theorem 9.2. Do not use this theorem to prove itself!
   Let $x \in S$ be arbitrary. We begin with double negation on $x \in S$ to get $\neg\neg x \in S$. Now, we apply addition to get $(\neg x \in U) \vee (\neg(\neg x \in S))$. We apply De Morgan's Law for propositions (Thm 2.11) to get $\neg(x \in U \wedge (\neg x \in S))$. Hence (by definition of complement), we get $\neg(x \in S^c)$. We now combine $x \in S$ with $S \subseteq U$ to get $x \in U$. By conjunction we get $x \in U \wedge \neg(x \in S^c)$. Finally (by definition of complement again), we get $x \in (S^c)^c$.

4. Let $R = \{x \in \mathbb{Z} : \exists y \in \mathbb{Z},\ x = 8y\}$, $S = \{x \in \mathbb{Z} : \exists y \in \mathbb{Z},\ x = 20y\}$, $T = \{x \in \mathbb{Z} : \exists y \in \mathbb{Z},\ x = 4y\}$. Prove or disprove that $R \cup S \subseteq T$.
   The statement is true. A correct proof must start with letting $x \in R \cup S$ be arbitrary. Then $x \in R \vee x \in S$. We now have two cases.
   Case $x \in R$: Hence there is $y \in \mathbb{Z}$ with $x = 8y$. We write $x = 4(2y)$, and $2y \in \mathbb{Z}$, so $x \in T$.
   Case $x \in S$: Hence there is $y \in \mathbb{Z}$ with $x = 20y$. We write $x = 4(5y)$, and $5y \in \mathbb{Z}$, so $x \in T$.
   Hence, in both cases, $x \in T$.

5. Let S, T be sets. Prove that $S \Delta T = T \Delta S$.
NOTE: This is part of Theorem 8.13. Do not use this theorem to prove itself!

Part 1 (proving $S\Delta T \subseteq T \Delta S$): Let $x \in S\Delta T$. Then $(x \in S \wedge x \notin T) \vee (x \notin S \wedge x \in T)$. By commutativity of $\vee, \wedge$ (Thm 2.8), we get $(x \in T \wedge x \notin S) \vee (x \notin T \wedge x \in S)$. Hence $x \in T\Delta S$.
Part 2 (proving $T\Delta S \subseteq S\Delta T$): Let $x \in T\Delta S$. Then $(x \in T \wedge x \notin S) \vee (x \notin T \wedge x \in S)$. By commutativity of $\vee, \wedge$ again, we get $(x \in S \wedge x \notin T) \vee (x \notin S \wedge x \in T)$. Hence $x \in S\Delta T$.

NOTE: Some of you used a variation of the definition, e.g. $(x \in S \wedge x \notin T) \vee (x \in T \wedge x \notin S)$. This was fine, provided you used the exact same version consistently throughout.

6. Find a set S such that $S \times (S \cap 2^S)$ is nonempty. Give S carefully, in list notation, and justify your answer.

Many solutions are possible. The key is to make $S \cap 2^S$ nonempty, i.e. for S to contain at least one of its own subsets. Notation and explanation are very important.
SOLUTION 1: Take $S = \{3, \{3\}\}$. Now $2^S = \{\emptyset, \{3\}, \{\{3\}\}, S\}$, so $S \cap 2^S = \{\{3\}\}$. Hence $S \times (S \cap 2^S) = \{(3, \{3\}), (\{3\}, \{3\})\}$, which is nonempty since it contains $(3, \{3\})$.
SOLUTION 2: Take $S = \{\emptyset\}$. Now $2^S = \{\emptyset, \{\emptyset\}\}$, so $S \cap 2^S = \{\emptyset\}$. Hence $S \times (S \cap 2^S) = \{(\emptyset, \emptyset)\}$, which is nonempty since it contains $(\emptyset, \emptyset)$.

7. Set $R = \{1, 2\}$, and $S = \mathbb{N}$. Prove or disprove that $|R \times S| = |S|$.
The statement is true, and to prove it we need a pairing between the elements of $R \times S$ and S. The natural one is: $(a, b) \leftrightarrow a + 2b - 2$. The first few pairings are:
$(1, 1) \leftrightarrow 1, (2, 1) \leftrightarrow 2, (1, 2) \leftrightarrow 3, (2, 2) \leftrightarrow 4, (1, 3) \leftrightarrow 5, (2, 3) \leftrightarrow 6, (1, 4) \leftrightarrow 7, \ldots$.

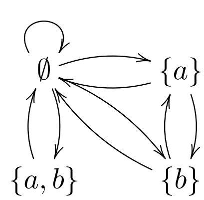
Some of you used this same pairing, but wanted to reverse the formula using cases.

$$n \leftrightarrow \begin{cases} (1, \lceil \frac{n}{2} \rceil) & n \text{ odd} \\ (2, \lceil \frac{n}{2} \rceil) & n \text{ even} \end{cases} \qquad 1 \leftrightarrow (1, 1), 2 \leftrightarrow (2, 1), 3 \leftrightarrow (1, 2), 4 \leftrightarrow (2, 2), \ldots$$

For the remaining problems 8-10, let $S = \{a, b\}$ and $T = 2^S$. Define relation R on T via $R = \{(x, y) : x \cap y = \emptyset\}$. Each of these problems has two parts.

8. Prove or disprove that R is symmetric. Also, prove or disprove that R is reflexive.
R is symmetric: Let $x, y \in T$ be arbitrary. Suppose that xRy. Then $x \cap y = \emptyset$. By commutativity of $\cap$ (Thm. 8.13), $y \cap x = x \cap y = \emptyset$. Hence yRx.
R is not reflexive: Take $x = \{a\} \in T$. We have $x \cap x = \{a\} \neq \emptyset$, so $(x, x) \notin R$.
(other counterexamples are possible)

9. Draw the digraph for relation R. Also, determine $|R|$.



We count directed edges to see that $|R| = 9$.

10. Prove or disprove that R is transitive. Also, prove or disprove that $R^{(2)} = T \times T$.
R is not transitive. We need a counterexample: $\{a, b\}R\emptyset$ and $\emptyset R\{a\}$ and $\{a, b\}\not R\{a\}$.

$R^{(2)} = T \times T$ is true. First, $R^{(2)} \subseteq T \times T$ is true since $R^{(2)} = R \circ R$ is a relation on T.
For the other direction, let $(x, y) \in T \times T$ be arbitrary. We have $x \cap \emptyset = \emptyset$, so $xR\emptyset$. We have $\emptyset \cap y = \emptyset$, so $\emptyset Ry$. Since $(x, \emptyset), (\emptyset, y) \in R$, we have $(x, y) \in R^{(2)}$.