MATH 245 S22, Exam 3 Solutions

1. Carefully state the following theorems: Associativity Theorem, Distributivity Theorem

   The Associativity Theorem states that, for any sets R, S, T, we have (a) $R \cap (S \cap T) = (R \cap S) \cap T$; and (b) $R \cup (S \cup T) = (R \cup S) \cup T$; and (c) $R \Delta (S \Delta T) = (R \Delta S) \Delta T$. The Distributivity Theorem states that, for any sets R, S, T, we have (a) $R \cap (S \cup T) = (R \cap S) \cup (R \cap T)$; and (b) $R \cup (S \cap T) = (R \cup S) \cap (R \cup T)$.

2. Carefully define the following terms: reflexive closure, $R^{(k)}$

   Given a set S and any relation R on S, the reflexive closure of R is defined as the relation $R \cup \{(x, x) : x \in S\}$. Given any set S and any relation R on S, we define $R^{(1)}$ as just R, and for $k \geq 2$, we define $R^{(k)} = R \circ R^{(k-1)}$.

3. Let S, T, U be sets with $S \subseteq U$ and $T \subseteq U$. Prove that $S \setminus T = S \cap T^c$.

   To prove two sets are equal we must prove that each is a subset of the other.
   (part 1) Let $x \in S \setminus T$. Then $x \in S \wedge x \notin T$. By simplification twice, $x \in S$ and $x \notin T$. Because $x \in S$ and $S \subseteq U$, we have $x \in U$. By conjunction, $x \in U \wedge x \notin T$; hence $x \in T^c$. By conjunction, $x \in S \wedge x \in T^c$; hence $x \in S \cap T^c$.
   (part 2) Let $x \in S \cap T^c$. Then $x \in S \wedge x \in T^c$. By simplification twice, $x \in S$ and $x \in T^c$. Hence $x \in U \wedge x \notin T$. By simplification, $x \notin T$. By conjunction, $x \in S \wedge x \notin T$; hence $x \in S \setminus T$.

4. Prove or disprove: For all sets S, T, we must have $S \setminus T \subseteq S \Delta T$.

   The statement is true. We can prove this directly, or with a theorem. We begin by letting S, T be arbitrary sets.

   (DIRECTLY) Let $x \in S \setminus T$. Then $x \in S \wedge x \notin T$. By addition, $(x \in S \wedge x \notin T) \vee (x \notin S \wedge x \in T)$. Hence $x \in S \Delta T$.

   (THEOREM) A in the book (Theorem 8.12(b)) states that $S \Delta T = (S \setminus T) \cup (T \setminus S)$. To finish we need to do exercise 8.14, proving that $A \subseteq A \cup B$ for all sets A, B. [Note: you may cite theorems but not exercises on exams.] The proof is: Let $x \in A$. By addition, $x \in A \vee x \in B$. Hence $x \in A \cup B$.

5. Find a set S such that $S \cap (S \times 2^S)$ is nonempty. Give S, carefully, in list notation.

   Many solutions are possible; it is important to have correct notation. One possible answer is $S = \{1, (1, \{1\})\}$. S contains two elements: the number 1 and the ordered pair $(1, \{1\})$. $(1, \{1\}) \in S$ because it is one of the two elements of S. Also $(1, \{1\}) \in S \times 2^S$ because it is an ordered pair whose first coordinate, namely 1, is an element of S and whose second coordinate, namely $\{1\}$, is an element of 2^S. Because $(1, \{1\})$ is in both S and $S \times 2^S$, it is in their intersection.

6. Let $S = \{x \in \mathbb{Z} : \exists y \in \mathbb{Z},\ x = 6y\}$ and $T = \{x \in \mathbb{Z} : \exists y \in \mathbb{Z},\ xy = 60\}$. Calculate, with justification, $|S \cap T|$.

We seek integers x in both S and in T. The first property means that $x = 6k$, for some integer k. The second means that $xy = 6ky = 60$, for some integer y. Hence $ky = 10$, so $k|10$. This gives solutions $k = \pm1, \pm2, \pm5, \pm10$; i.e. $S \cap T = \{6, -6, 12, -12, 30, -30, 60, -60\}$ so $|S \cap T| = 8$.

7. Let A, B, C be sets with $A \setminus B \subseteq C$. Prove that $A \subseteq B \cup C$.

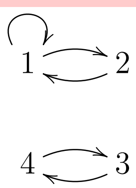
Let $x \in A$. There are now two cases, depending on whether $x \in B$ or $x \notin B$.
Case $x \in B$: By addition, $x \in B \vee x \in C$, so $x \in B \cup C$.
Case $x \notin B$: By conjunction, $x \in A \wedge x \notin B$, so $x \in A \setminus B$. Because $A \setminus B \subseteq C$, we have $x \in C$. By addition, $x \in B \vee x \in C$, so $x \in B \cup C$.
In both cases, $x \in B \cup C$.

8. Let S, T be sets and R_1, R_2 relations from S to T. Suppose that $R_1 \subseteq R_2^{-1}$ and $R_2 \subseteq R_1^{-1}$. Prove that $R_1 = R_2^{-1}$.

We have $R_1 \subseteq R_2^{-1}$ by hypothesis, to it only remains to prove that $R_2^{-1} \subseteq R_1$.
Let $x \in R_2^{-1}$ be arbitrary. Then $x = (b, a)$, where $(a, b) \in R_2$. Because $R_2 \subseteq R_1^{-1}$, we have $(a, b) \in R_1^{-1}$. Thus $(b, a) \in R_1$. But that's just x, so we have proved $x \in R_1$.

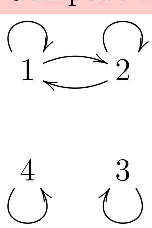
For problems 9,10:
Let $A = \{1, 2, 3, 4\}$ and take $R = \{(1, 1), (1, 2), (2, 1), (3, 4), (4, 3)\}$, a relation on A.

9. Draw the digraph representing R. Determine, with justification, whether or not R is each of: reflexive, symmetric, and transitive.



R is not reflexive because, e.g., $(2, 2) \notin R$.
R is symmetric because for every pair of vertices, either both directed edges or neither are present.
R is not transitive because, e.g., $(3, 4), (4, 3) \in R$ and $(3, 3) \notin R$.

10. Compute $R \circ R$. Give your answer both as a digraph and as a set.



$R \circ R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)\}$.
Note: every missing or extra piece in a solution, will cost points.