MATH 245 S21, Final Exam

(120 minutes, open book, open notes)

1. Exam instructions.
2. (10 pts) What is the category of "$R \subseteq S$"? Select the one best answer.
(i) proposition; (ii) set; (iii) element; (iv) number; (v) inequality.
3. (10 pts) What is the category of $2^{(2^{\mathbb{Z}})}$? Select the one best answer.
(i) proposition; (ii) integer(s); (iii) set of integer(s); (iv) set of sets of integer(s); (v) set of sets of sets of integer(s).
4. (10 pts) Consider $R = \{(a,c),(a,a),(c,b)\}$, a relation on $S = \{a,b,c\}$. Select which of the following properties R satisfies. (you may select as many as you wish, including none or all).
(i) reflexive; (ii) irreflexive; (iii) transitive; (iv) left-total; (v) left-definite.
5. (10 pts) Let R_1 be the divisibility relation $|$ on $\mathbb{N}$, and let R_2 be the usual order on $\mathbb{N}$. Let R denote the lex order on $\mathbb{N} \times \mathbb{N}$. Select which of the following are true. (you may select as many as you wish, including none or all).
(i) $(2,4)R(4,2)$; (ii) $(4,2)R(2,4)$; (iii) $(2,4)R(3,2)$; (iv) $(3,2)R(2,4)$; (v) $(2,4)\|(3,2)$.
6. (20 pts) Let A, B, S, T be sets. Suppose that $A \subseteq B \subseteq S \subseteq T \subseteq A$. Prove that $B = T$.
7. (20 pts) Let S be a set, and R a relation on S. Prove or disprove: $R\Delta(S \times S) = R^c$.
8. (20 pts) Let S be a set. Prove that the diagonal relation $R_{diagonal}$ on S is an equivalence relation.
9. (20 pts) Consider the equivalence relation $\equiv$, modulo 7, on $\mathbb{Z}$. Prove that, for all integers a, b, the two sets $[a+b]$ and $\{x+y : x \in [a], y \in [b]\}$ are equal.
10. (20 pts) Prove or disprove: $\forall x \in \mathbb{Q},\ !y \in \mathbb{Q},\ |x-y| = |y|$.
11. (20 pts) Prove or disprove: $\forall x \in \mathbb{Q},\ \forall y \in \mathbb{Q},\ \exists z \in \mathbb{Q},\ (x < y) \rightarrow (x < z < y)$.
12. (20 pts) Prove or disprove: $\forall x \in \mathbb{R}$, if x is irrational, then $\frac{1}{x}$ is irrational.
13. (20 pts) Let $a, b \in \mathbb{Z}$ with $a > b \geq 0$. Use some form of induction to prove:
$\forall i \in \mathbb{N},\ \binom{a+i}{b+i} > \binom{a}{b}$.
14. (20 pts) Consider the divisibility partial order $|$ on $\mathbb{N}$. Find $a, b \in \mathbb{N}$ so that the interval poset $[a,b]$ has height 5 and width 2. Be sure to justify your answer.
15. (20 pts) Prove or disprove: For every relation R on $\mathbb{Z}$, if R is a function then $R \circ R$ is a function.
16. (20 pts) Prove or disprove: For every relation R on $\mathbb{Z}$, if $R \circ R$ is a function then R is a function.
17. (20 pts) Consider the relation $R = \{(x,y) : x(x+1)|y(y+1)\}$ on $\mathbb{N}$. Prove that this is a partial order.
18. (20 pts) Consider the partial order $R = \{(x,y) : x(x+1)|y(y+1)\}$ on $\{1,2,3,4,5,6\}$. Draw the Hasse diagram, and identify all maximal, greatest, minimal, and least elements.