MATH 245 S19, Exam 3 Solutions

1. Carefully define the following terms: = (for sets), union, disjoint.

   Two sets are equal if they contain exactly the same elements. Given sets S, T, their union is the set $\{x : x \in S \vee x \in T\}$. Two sets are disjoint if their intersection is equal to the empty set.

2. Carefully define the following terms: De Morgan's Law (for sets), Cantor's Theorem, transitive

   Given sets S, T, U with $S \subseteq U$ and $T \subseteq U$, De Morgan's law states that (a) $(S \cup T)^c = S^c \cap T^c$; and (b) $(S \cap T)^c = S^c \cup T^c$. Cantor's Theorem states that no set is equicardinal with its power set. Given a relation R on set S, we say that R is transitive if for all $x, y, z \in S$, $(xRy \wedge yRz) \to xRz$.

3. Let R, S be sets with $R \setminus S = S \setminus R$. Prove that $R \subseteq S$.

   Let $x \in R$ be arbitrary. We will prove that $x \in S$ by contradiction; that is, assume that $x \notin S$. By conjunction, $(x \in R) \wedge (x \notin S)$. Hence, $x \in R \setminus S$. Because $R \setminus S = S \setminus R$, in fact $x \in S \setminus R$. Hence $x \in S \wedge x \notin R$. By simplification, $x \notin R$. This is a contradiction. Hence, $x \in S$.

4. Prove or disprove: For all sets R, S, T satisfying $R \subseteq S$, $S \subseteq T$, and $T \subseteq R$, we must have $R = S$.

   The statement is true. We have $R \subseteq S$ by hypothesis, so it suffices to prove that $S \subseteq R$. Let $x \in S$ be arbitrary. Because $S \subseteq T$, we have $x \in T$. Because $T \subseteq R$, $x \in R$.

5. Prove or disprove: For all sets R, S, we have $R \times S = S \times R$.

   The statement is false; to disprove, we need explicit examples for R, S. One possible answer is $R = \{a\}, S = \{b\}$. To prove that $R \times S \neq S \times R$, we need an explicit element that is in one set but not the other. $(a, b) \in R \times S$, but $(a, b) \notin S \times R = \{(b, a)\}$.

6. Prove or disprove: For all sets R, S, we have $|R \times S| = |S \times R|$.

   Note: The theorem $|R \times S| = |R| \cdot |S|$ holds only for *finite* sets R, S and will only provide partial credit.
   To prove two sets are equicardinal, we need an explicit pairing between their elements. The natural one is $(x, y) \leftrightarrow (y, x)$, for every $x \in R$ and $y \in S$.

7. Consider relation $R = \{(a, b) : a^2 \geq b\}$ on $\mathbb{Q}$. Prove or disprove that R is reflexive.

   The statement is false; to disprove, we need an explicit counterexample. If we take $a = b = \frac{1}{2}$, we see that $a^2 = \frac{1}{4} \not\geq \frac{1}{2} = b$, so $(\frac{1}{2}, \frac{1}{2}) \notin R$ and hence R is not reflexive.

8. Prove or disprove: For all sets R, S, we have $2^R \cup 2^S = 2^{R\cup S}$.

The statement is false; to disprove, we need explicit examples for R, S. One possible answer is $R = \{a\}, S = \{b, c\}$. To prove that $2^R \cup 2^S \neq 2^{R\cup S}$, we need an explicit element that is in one set but not the other. $\{a, b\} \in 2^{R\cup S}$, as it is a subset of $R \cup S = \{a, b, c\}$. However, $\{a, b\} \notin 2^R$, as it is not a subset of R. $\{a, b\} \notin 2^S$, as it is not a subset of S. Hence, $\{a, b\} \notin 2^R \cup 2^S$.

9. Let R, S, T be sets. Prove that $R \cap (S \cup T) \subseteq (R \cap S) \cup (R \cap T)$. Your answer should not use any theorems about sets.

SOLUTION 1: Let $x \in R \cap (S \cup T)$. Hence $x \in R \wedge x \in (S \cup T)$. By simplification twice, we get $x \in R$ and $x \in (S \cup T)$. Hence, $x \in S \vee x \in T$. We now have two cases:
Case $x \in S$: By conjunction, $x \in R \wedge x \in S$. Hence, $x \in (R \cap S)$. By addition, $x \in (R \cap S) \vee x \in (R \cap T)$.
Case $x \in T$: By conjunction, $x \in R \wedge x \in T$. Hence, $x \in (R \cap T)$. By addition, $x \in (R \cap S) \vee x \in (R \cap T)$.
In both cases, $x \in (R \cap S) \vee x \in (R \cap T)$, and hence $x \in (R \cap S) \cup (R \cap T)$.

SOLUTION 2: Let $x \in R \cap (S \cup T)$. Hence $x \in R \wedge x \in (S \cup T)$. Hence $(x \in R) \wedge (x \in S \vee x \in T)$. Applying the distributivity theorem (for propositions), we get $(x \in R \wedge x \in S) \vee (x \in R \wedge x \in T)$. Hence $(x \in R \cap S) \vee (x \in R \cap T)$, and finally $x \in (R \cap S) \cup (R \cap T)$.

10. Consider relation $R = \{(a, b) : b \leq a \leq 3b\}$ on $\mathbb{N}_0$. Compute and simplify $R \circ R$. Your answer should not use quantifiers.

We start with $R \circ R = \{(a, c) : \exists b \in \mathbb{N}_0, aRb \wedge bRc\}$. This gives us four inequalities: $b \leq a \leq 3b$ and $c \leq b \leq 3c$. We combine two of them as $c \leq b \leq a$, and the other two as $a \leq 3b \leq 9c$. Hence the simplified version is $R \circ R = \{(a, c) : c \leq a \leq 9c\}$. Finding this, with justification, is enough for full credit.

For anyone curious about a proof that these sets are equal, here is an explicit calculation of b: Let (a, c) satisfy $c \leq a \leq 9c$. If $c \leq a \leq 3c$, we take $b = c$. If instead $3c < a \leq 9c$, we take $b = 3c$.