MATH 245 S19, Exam 1 Solutions

1. Carefully define the following terms: even, tautology, converse, predicate.

   An integer n is even if there exists an integer m, with $n = 2m$. A (compound) proposition is a tautology if it is logically equivalent to T. The converse of conditional proposition $p \to q$ is $q \to p$. A predicate is a collection of propositions, indexed by one or more free variables, each drawn from its domain.

2. Carefully define the following terms: Division Algorithm theorem, Commutativity theorem, Conjunction semantic theorem, Contrapositive Proof theorem.

   The Division Algorithm theorem states that for any $a, b \in \mathbb{Z}$, with $b \geq 1$, there are unique integers q, r with $a = bq + r$ and $0 \leq r < b$. The Commutativity theorem states that if p, q are propositions, then $p \wedge q \equiv q \wedge p$ and $p \vee q \equiv q \vee p$. The Conjunction semantic theorem states if p, q are propositions, then $p, q \vdash p \wedge q$. The Contrapositive Proof theorem states that if $\neg q \vdash \neg p$ is valid, then $p \to q$ is true.

3. Let $n \in \mathbb{N}$ be arbitrary. Prove that $n|n!$.

   Since $n \geq 1$, $n! = n \cdot (n-1)!$. Since $(n-1)!$ is an integer, $n|n!$.

4. Let $a, b, c \in \mathbb{Z}$. Suppose that $a \leq b$. Prove that $a + c \leq b + c$.
   Note: do not just cite a theorem.

   Since $a \leq b$, $b - a \in \mathbb{N}_0$. Hence $(b + c) - (a + c) = b - a \in \mathbb{N}_0$, and hence $a + c \leq b + c$.
   Note: Solutions need to use the definition of $\leq$, twice.

5. Let p, q be propositions. Prove that $p \uparrow q \equiv \neg(p \wedge q)$.

   Pf. The third and fifth columns of the truth table (to the right) agree; hence the two propositions are equivalent.

p	q	$p \uparrow q$	$p \wedge q$	$\neg(p \wedge q)$
T	T	F	T	F
T	F	T	F	T
F	T	T	F	T
F	F	T	F	T

6. Prove or disprove: $\forall x \in \mathbb{R}, x^2 \geq x$.

   The statement is false. We need one explicit counterexample. Take $x = \frac{1}{2} \in \mathbb{R}$. We have $x^2 = \frac{1}{4} \not\geq \frac{1}{2} = x$.

7. Prove or disprove: For arbitrary $x \in \mathbb{R}$, if x is irrational then $2x - 1$ is irrational.

   The statement is true. Contrapositive proof. We assume $2x - 1$ is rationsl. Hence there are integers a, b, with $b \neq 0$, such that $2x - 1 = \frac{a}{b}$. Now, $2x = \frac{a}{b} + 1 = \frac{a+b}{b}$, and $x = \frac{a+b}{2b}$. We have $a + b, 2b \in \mathbb{Z}$, and $2b \neq 0$, so x is rational.

8. Without using truth tables, prove the Composition Theorem: $(p \to q) \wedge (p \to r) \vdash p \to (q \wedge r)$.

   METHOD 1: direct proof. We apply Conditional Interpretation twice to the hypothesis, to get $((\neg p) \vee q) \wedge ((\neg p) \vee r)$. Now we apply distributivity to get $(\neg p) \vee (q \wedge r)$. We apply Conditional Interpretation again to get $p \to (q \wedge r)$.

   METHOD 2: cases, based on p. Case p is false: By addition, $(q \wedge r) \vee \neg p$.
   Case p is true: By simplification on the hypothesis, $p \to q$; and, by modus ponens, q. Now by simplification on the hypothesis the other way, $p \to r$; and, by modus ponens, r. Now, by conjunction, $q \wedge r$. By addition, $(q \wedge r) \vee \neg p$. Hence, in both cases, $(q \wedge r) \vee \neg p$. We end with conditional interpretation, giving $p \to (q \wedge r)$.

9. State and prove modus tollens, using semantic theorems only (no truth tables).

   Thm: Let p, q be propositions. Then $p \to q, \neg q \vdash \neg p$.
   Pf 1: We assume $p \to q$ and $\neg q$. By conditional interpretation, $q \vee \neg p$. By disjunctive syllogism, $\neg p$.
   Pf 2: We assume $p \to q$ and $\neg q$. We have $p \to q \equiv (\neg q) \to (\neg p)$, its contrapositive. By modus ponens, $\neg p$.

10. Prove or disprove: $\exists x \in \mathbb{R}\ \forall y \in \mathbb{R}, |y| \leq |y + x|$.

    The statement is true. Take $x = 0$. Now, let $y \in \mathbb{R}$ be arbitrary. $|y| = |y + 0| \leq |y + 0| = |y + x|$.
    Note: For full credit, the structure must be: (1) specific choice for x; (2) let y be arbitrary; (3) algebra; (4) ends with $|y| \leq |y + x|$. Also, a solution must specify whether you are proving or disproving.