MATH 245 F18, Exam 1 Solutions

1. Carefully define the following terms: floor, divides, nand, Commutativity theorem (for propositions).

Let $x \in \mathbb{R}$. Then there is a unique integer n, which we call the floor of x, which satisfies $n \le x < n+1$. Let $a, b \in \mathbb{Z}$. We say that a divides b if there exists some $c \in \mathbb{Z}$ with $ac = b$. Let p, q be propositions. p nand q is a compound proposition that is F if p, q are both T, and T otherwise. The Commutativity theorem states that for any propositions p, q, $p \vee q \equiv q \vee p$ and $p \wedge q \equiv q \wedge p$.

2. Carefully define the following terms: Double Negation semantic theorem, Vacuous Proof theorem, converse, predicate.

The Double Negation semantic theorem states that for any proposition p, $\neg(\neg p) \equiv p$. The Vacuous Proof theorem states that for any propositions p, q, $\neg p \vdash p \to q$. The converse of conditional proposition $p \to q$ is $q \to p$. A predicate is a collection of propositions, indexed by one or more free variables, each drawn from its domain.

3. Calculate, and simplify, $\binom{100}{1} - \binom{100}{0}$.

We have $\binom{100}{1} = \frac{100!}{99!1!} = \frac{100 \cdot 99!}{99!1!} = \frac{100}{1!} = \frac{100}{1} = 100$, cancelling 99! numerator and denominator. We also have $\binom{100}{0} = \frac{100!}{100!0!} = \frac{1}{0!} = \frac{1}{1} = 1$, cancelling 100! numerator and denominator. Subtracting, we get $100 - 1 = 99$.

4. Let $a, b \in \mathbb{Z}$, with $a \le b$. Prove that $a + 1 \le b + 2$, without using any theorems.

Because $a \le b$, the integer $b - a \in \mathbb{N}_0$. We also know that $1 \in \mathbb{N}_0$, and their sum $b - a + 1 \in \mathbb{N}_0$. But also $b - a + 1 = (b+2) - (a+1)$, so $a + 1 \le b + 2$.

5. State the Conditional Interpretation Theorem, and prove it using a truth table.

Thm. Let p, q be propositions. Then $p \to q \equiv q \vee \neg p$.

Pf. The third and fifth columns of the truth table (to the right) agree; hence the two propositions are equivalent.

p	q	$p \to q$	$\neg p$	$q \vee \neg p$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

6. Fix our domain to be $\mathbb{R}$. Simplify the proposition $\neg(\forall x\ \exists y\ \forall z,\ x \le y < z)$ as much as possible (where nothing is negated).

We begin by pulling $\neg$ into the quantifiers, as $\exists x\ \forall y\ \exists z\ \neg(x \le y < z)$. Note that $x \le y < z \equiv (x \le y) \wedge (y < z)$, so we apply De Morgan's law to get $\exists x\ \forall y\ \exists z\ (\neg(x \le y)) \vee \neg(y < z)$. Lastly, we simplify the inequalities to get $\exists x\ \forall y\ \exists z\ (x > y) \vee (y \ge z)$. Note that this can NOT be written as a double inequality.

7. Let $x \in \mathbb{R}$. Suppose that x is not odd. Prove that $\frac{x}{3}$ is not odd.

Warning: A direct proof is not recommended, because "not odd" does not imply "even" for real numbers.

We use a contrapositive proof. Assume that $\frac{x}{3}$ is not not odd, i.e. odd. Hence $\frac{x}{3}$ is an integer, and there is some integer n with $\frac{x}{3} = 2n + 1$. Multiplying both sides by 3, we have $x = 3(2n+1) = 2(3n) + 3 = 2(3n+1) + 1$. Since $3n + 1$ is an integer, x is odd, and hence not not odd.

8. Without using truth tables, prove the Composition Theorem: $(p \to q) \wedge (p \to r) \vdash p \to (q \wedge r)$.

We use a direct proof. We apply Conditional Interpretation twice to the hypothesis, to get $((\neg p) \vee q) \wedge ((\neg p) \vee r)$. Now we apply distributivity to get $(\neg p) \vee (q \wedge r)$. We apply Conditional Interpretation again to get $p \to (q \wedge r)$.

9. Simplify $\neg((p \to q) \wedge (\neg q))$ as much as possible (where only basic propositions are negated).

METHOD 1: We apply Conditional Interpretation to get $\neg((q \vee \neg p) \wedge (\neg q))$, and distributivity to get $\neg((q \wedge \neg q) \vee ((\neg p) \wedge (\neg q)))$. Because $q \wedge \neg q \equiv F$, and $F \vee r \equiv r$ (for $r = ((\neg p) \wedge (\neg q))$), this simplifies as $\neg((\neg p) \wedge (\neg q))$. Applying De Morgan's Law, we get $(\neg\neg p) \vee (\neg\neg q)$. Finally, applying Double Negation twice, we get $p \vee q$.

METHOD 2: We start with De Morgan's Law, getting $(\neg(p \to q)) \vee (\neg\neg q)$. We apply Double negation, getting $(\neg(p \to q)) \vee q$. We apply Conditional Interpretation, getting $(\neg(q \vee \neg p)) \vee q$. We apply De Morgan's Law and Double Negation, getting $((\neg q) \wedge p) \vee q$. We apply distributivity, getting $((\neg q) \vee q) \wedge (p \vee q)$. Since $(\neg q) \vee q \equiv T$, and $T \wedge r \equiv r$ (for $r = (p \vee q)$), the final result is $p \vee q$.

10. Fix our domain to be $\mathbb{R}$. Prove or disprove: $\forall x\ \exists y\ \forall z,\ x^2 \le y^2 + z^2$.

The statement is true. Let $x \in \mathbb{R}$ be arbitrary. We will choose $y = x$. Now, let $z \in \mathbb{R}$ be arbitrary. We have $z^2 \ge 0$, a property of squares. We now add x^2 to both sides, getting $z^2 + x^2 \ge 0 + x^2 = x^2$. Finally, since $y = x$, also $y^2 = x^2$, so $z^2 + y^2 \ge x^2$.