Polygonal Simplex Numbers

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Polygonal numbers $P_{s,n}$ can be arranged in a figure with s sides and n dots per side, recursively containing the arrangement of $P_{s,n-1}$ like Russian nesting dolls. Familiar triangular and square numbers are $P_{3,n}$ and $P_{4,n}$, respectively. We consider the set of all nontrivial polygonal numbers, as a subset of the naturals. Allowing $s \leq 2$ or $n \leq 2$ trivializes this question, since then the two sets coincide. This set of nontrivial polygonal numbers $\mathcal{P}$ (see [1]) was recently proved in [6] to contain every cube, and in [2] to contain almost every perfect power (apart from 2^p, for p prime). We continue these investigations by considering a different natural set of figurate numbers, the r-simplex numbers $P_r(k)$, i.e. those which can be arranged to form an r-simplex with k dots per side. We ask which r-simplex numbers are polygonal. Note that 1-simplex numbers are line segments and 2-simplex numbers are triangular, so we skip past these trivialities and focus on $r \geq 3$ and $k \geq 3$.

We begin by recalling two key formulas, for the polygonal and simplex numbers respectively:

$$\mathcal{P} = \left\{ P_{s,n} = \frac{(s-2)n^2 - (s-4)n}{2} : s \geq 3, n \geq 3 \right\}, \text{ and}$$

$$P_r(k) = \binom{k+r-1}{r} = \frac{k(k+1)(k+2)\cdots(k+r-1)}{r!}.$$

We will need a theorem proved by Edouard Lucas in a series of papers published in 1878 [3, 4, 5]:

Theorem 1 (Lucas's Theorem)**.** *Let p be prime and let $a, b \in \mathbb{N}$. Express a, b in base p as $a = (a_j a_{j-1} \cdots a_1 a_0)_p, b = (b_j b_{j-1} \cdots b_1 b_0)_p$. Then*

$$\binom{a}{b} \equiv \prod_{i=0}^{j} \binom{a_i}{b_i} \pmod{p},$$

where we make the standard interpretation of $\binom{a_i}{b_i} = 0$ if $a_i < b_i$.

More specifically, we will need a corollary of Lucas's Theorem.

Corollary 2. *Let p be prime and let $a, b \in \mathbb{N}$. Express a, b in base p as $a = (a_j a_{j-1} \cdots a_1 a_0)_p, b = (b_j b_{j-1} \cdots b_1 b_0)_p$. Then the following are equivalent:*

1. *p divides* $\binom{a}{b}$;

2. *There is some index* $i \in [0, j]$ *where* $a_i < b_j$.

It turns out that many r-simplex numbers are polygonal of the particular type $P_{s,3}$. These are characterized by our first theorem. Note that the characterization aligns well with Corollary 2.

Theorem 3. *Let* $r, k \in \mathbb{N}$ *with* $r, k \geq 3$. *Then there is some* $s \geq 3$ *with* $P_r(k) = P_{s,3}$ *if and only if* 3 *divides* $\binom{k+r-1}{r}$.

Proof. We begin by noting that $P_{s,3} = \frac{(s-2)3^2-(s-4)3}{2} = 3s - 3$. Since we only allow $s \geq 3$, we see that $P_{s,3}$ takes on all multiples of 3 that are 6 or greater. Recall that $P_r(k) = \binom{k+r-1}{r}$. If $k \geq 3$, then $P_r(k) \geq \binom{r+2}{r} \geq 6$. Therefore it is of the type $P_{s,3}$ if and only if 3 divides $P_r(k)$. □

The following result gives infinitely many r for which at least two thirds of k's give a polygonal $P_r(k)$.

Theorem 4. *Let* $r, k \in \mathbb{N}$ *with* $r, k \geq 3$, $r \equiv 2 \pmod 3$ *and* $k \not\equiv 1 \pmod 3$. *Then* $P_r(k)$ *is polygonal.*

Proof. We convert to base 3 as $r = (r_j \cdots r_1 r_0)_3$ and $k + r - 1 = (a_j \cdots a_1 a_0)_3$. Note that $r_0 = 2$ since $r \equiv 2$. Now, $k \not\equiv 1$ and $r \equiv 2 \pmod 3$, so $k + r - 1 \not\equiv 2$, so $a_0 < 2$. Applying Corollary 2, we find that $a_0 < r_0$, so 3 divides $\binom{k+r-1}{r}$. Applying Theorem 3, we find that there is some $s \geq 3$ with $P_{r,k} = P_{s,3}$. □

Theorem 4 is powerful for one third of possible r's. Theorem 5 is half as powerful for half of the remaining r's.

Theorem 5. *Let* $r, k \in \mathbb{N}$ *with* $r, k \geq 3$, $r \equiv 1 \pmod 3$, *and* $k \equiv 0 \pmod 3$. *Then* $P_r(k)$ *is polygonal.*

Proof. We convert to base 3 as $r = (r_j \cdots r_1 r_0)_3$ and $k + r - 1 = (a_j \cdots a_1 a_0)_3$. Note that $r_0 = 1$ since $r \equiv 1$. Now, since $k \equiv 0$ and $r \equiv 1$, we have $k + r - 1 \equiv 0$, so $a_0 = 0$. We now apply Corollary 2 and Theorem 3 as before. □

Theorems 4 and 5 only looked at the terminal ternary digits. We can use similar methods to find many such theorems, but there are diminishing returns as we progress. It may seem at first glance that Theorem 6 covers $(3/9)(4/9) = 12/81 = 4/27$ of all cases, but five of them[1] duplicate results from Theorems 4 and 5. Hence only $7/81$ of all remaining cases are covered by Theorem 6.

Theorem 6. *Let* $r, k \in \mathbb{N}$ *with* $r, k \geq 3$. *Suppose that* $r \mod 9 \in \{6, 7, 8\}$, *while* $k \mod 9 \in \{3, 4, 5, 6\}$. *Then* $P_r(k)$ *is polygonal.*

Proof. We convert to base 3 as $r = (r_j \cdots r_1 r_0)_3$ and $k + r - 1 = (a_j \cdots a_1 a_0)_3$. Note that $r_1 = 2$ by hypothesis. We see that $k + r - 1 \in \{9, 10, \ldots, 14\}$, so $k + r - 1 \mod 9 \in \{0, 1, 2, 3, 4, 5\}$, and so $a_1 \in \{0, 1\}$. We now apply Corollary 2 and Theorem 3 as before. □

[1] $(r \equiv 8, k \equiv 3), (r \equiv 8, k \equiv 5), (r \equiv 8, k \equiv 6), (r \equiv 7, k \equiv 3), (r \equiv 7, k \equiv 6)$

We could continue finding similar theorems for $n = 3$, but instead we observe that there are analogues to Theorem 3 for other values of n. One example follows.

Theorem 7. *Let $r, k \in \mathbb{N}$ with $r, k \geq 3$. Then there is some $s \geq 3$ with $P_r(k) = P_{s,5}$ if and only if 5 divides $\binom{k+r-1}{r}$ and 2 does not.*

Proof. We begin by noting that $P_{s,5} = \frac{(s-2)5^2-(s-4)5}{2} = 10s - 15$. Since we only allow $s \geq 3$, we see that $P_{s,3}$ takes on all odd multiples of 5 that are greater than 5, i.e. $15, 25, 35, \ldots$. Recall that $P_r(k) = \binom{k+r-1}{r}$. If $k \geq 3$, then $P_r(k) \geq \binom{r+2}{r} \geq 6$. Therefore it is of the type $P_{s,3}$ if and only if $P_r(k) \equiv 5 \pmod{10}$. By the Chinese Remainder Theorem, this holds if and only if $P_r(k) \equiv 0 \pmod 5$ and $P_r(k) \equiv 1 \pmod 2$. $\square$

Theorem 8 uses Theorem 7 to prove that at least 1/5 of $P_r(3)$ are of the type $P_{s,5}$. However, one third of these are included in Theorem 5, so this only improves things by $2/15 \approx 13\%$. These are the sort of diminishing returns that set in with further theorems along these lines.

Theorem 8. *Let $r \in \mathbb{N}$ with $r \geq 3$ and $r \mod 20 \in \{4, 8, 9, 13\}$. Then $P_r(3)$ is polygonal.*

Proof. Note that the hypothesis implies that $r \mod 5 \in \{3, 4\}$ and $r \mod 4 \in \{0, 1\}$. We calculate $P_r(3) = \binom{2+r}{r} = \frac{(r+2)(r+1)}{2}$. If $r = 4q$ for some integer q, then $P_r(3) = \frac{(4q+2)(4q+1)}{2} = (2q+1)(4q+1)$, which is odd. If instead $r = 4q+1$ for some integer q, then $P_r(3) = \frac{(4q+3)(4q+2)}{2} = (4q+3)(2q+1)$, which is also odd. Hence either way $\binom{k+r-1}{r}$ is odd. Now we write in base 5 as $r = (r_j \cdots r_1 r_0)_5$ and $k + r - 1 = (a_j \cdots a_1 a_0)_5$. Since $r \mod 5 \in \{3, 4\}$, we have $r_0 \in \{3, 4\}$. We calculate $k + r - 1 = 2 + r$, so $k + r - 1 \mod 5 \in \{0, 1\}$ and hence $a_0 \in \{0, 1\}$, so $a_0 < r_0$. Hence, by Corollary 2, $\binom{k+r-1}{r}$ is a multiple of 5, and by Theorem 7, $P_r(3)$ is polygonal. $\square$

We now turn to an asymptotic result. By looking at all of the digits, we can prove that $P_r(k)$ is polygonal (just with $n = 3$) with high probability.

Theorem 9. *Let $N \in \mathbb{N}$, and let r, k each be chosen independently, uniformly at random, from $[0, 3^N)$. Then $P_r(k)$ is polygonal with probability at least $(1 - (1/3)^{N-1})^2(1 - (2/3)^N) \geq 1 - 3(2/3)^N$.*

Proof. Each of r, k is at least 3 with probability $1 - 3^{N-1}$. Condition on this assumption. Now, each ternary digit of r, k is equally likely to be $0, 1, 2$, so each ternary digit of $k + r - 1, r$ are equally likely to be $0, 1, 2$. Of these nine cases, three – $(1, 2), (0, 1), (0, 2)$ – guarantee that $P_r(k)$ is polygonal via Corollary 2 and Theorem 3. To fail this test with all N digits is of probability $(2/3)^N$, so to pass it is of probability $1 - (2/3)^N$. Finally, $(1 - (1/3)^{N-1})^2(1 - (2/3)^N) = 1 - (2/3)^N - 2(1/3)^{N-1} + (1/3)^{N-1}(2(2/3)^N + (1/3)^{N-1}(1 - (2/3)^N)) \geq 1 - (2/3)^N - 2(1/3)^{N-1} \geq 1 - (1 + 6/2^N)(2/3)^N \geq 1 - 3(2/3)^N$, where the final inequality holds for all $N \geq 2$ (and we check the trivial $N = 1$ separately). $\square$

Our last two results are specifically for small r: $r = 3$ are the tetrahedral numbers and $r = 4$ are the pentatope numbers. The nicer result is for the latter.

Theorem 10. *Let $k \geq 3$. Then the pentatope number $P_4(k)$ is polygonal.*

Proof. If $k \equiv 0 \pmod 3$, then Theorem 5 shows that $P_4(k)$ is polygonal, with $n = 3$. Otherwise $P_4(k)$ will be polygonal with $s = 5$. We set $q = \frac{(k+1)(k+2)}{6}$. Note that q is an integer exactly when $k \not\equiv 0 \pmod 3$. We directly calculate $P_{5,q} = \frac{3q^2-q}{2} = \frac{k^4}{24} + \frac{k^3}{4} + \frac{11k^2}{24} + \frac{k}{4} = \frac{k(k+1)(k+2)(k+3)}{24} = P_4(k)$. □

Lastly, we consider tetrahedral numbers. Neither Theorem 4 nor Theorem 5 help here, unfortunately. However, we find that almost all tetrahedral numbers are polygonal. The cases of Theorem 11 may seem strange and random, but in fact there are often few choices of (n, s) that work, or only one!

Theorem 11. *Let $k \geq 3$. If $k \not\equiv 6 \pmod{18}$, then $P_3(k)$ is polygonal.*

Proof. The proof proceeds in five cases. In each case, just as in Theorem 10, we explicitly compute $P_{s,n}$ and $P_3(k)$ to see they are equal.
Case $k \equiv 2 \pmod 3$: Take $n = k, s = \frac{k+10}{3}$.
Case $k \equiv 1 \pmod 3$: Take $n = \frac{k+2}{3}, s = 3k + 8$.
Case $k \equiv 0 \pmod 9$: Take $n = 3, s = 1 + \frac{k(k+1)(k+2)}{18}$.
Case $k \equiv \pm 3 \pmod{18}$: Take $n = \frac{k+1}{2}, s = 6 + \frac{4k}{3}$.
Case $k \equiv 12 \pmod{18}$: Take $n = 4, s = \frac{48+k(k+1)(k+2)}{36}$. □

The "missing" tetrahedral case of $k \equiv 6 \pmod{18}$ really is missing: many of those are computed to not be polygonal, in a complicated pattern.

In fact, these many positive results may give the impression that almost all simplex numbers are polygonal, but this is just due to a difficulty in proving negative results. We conjecture that there are infinitely many simplex numbers that fail to be polygonal. Even more strongly, we conjecture that there are infinitely many tetrahedral numbers that fail to be polygonal.

These conjectures are supported with some experimental evidence. The first five simplex numbers that fail to be polygonal are $20, 56, 2600, 6188, 13244$. The quantity of simplex numbers that fail to be polygonal, up to various thresholds, are summarized in the table below.

$\leq 10^2$	$\leq 10^3$	$\leq 10^4$	$\leq 10^5$	$\leq 10^6$	$\leq 10^7$	$\leq 10^8$	$\leq 10^9$	$\leq 10^{10}$
2	2	4	9	16	24	40	75	151

References

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