Polygonal Simplex Numbers

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Polygonal numbers $P_{s,n}$ can be arranged in a figure with s sides and n dots per side, recursively containing the arrangement of $P_{s,n-1}$ like Russian nesting dolls. The familiar triangular and square numbers are $P_{3,n}$ and $P_{4,n}$, respectively. We consider the set of all nontrivial polygonal numbers, as a subset of the naturals. Allowing $s \leq 2$ or $n \leq 2$ trivializes this question, since then the two sets coincide. This set of nontrivial polygonal numbers $\mathcal{P}$ (see [1]) was recently proved in [6] to contain every cube, and in [2] to contain almost every perfect power (apart from 2^p, for p a prime).

We continue these investigations by considering a different natural set of figurate numbers, the r-simplex[1] numbers $S_r(k)$, i.e. those which can be arranged to form an r-simplex (the r-dimensional generalization of a triangle) with k dots per side. We ask which r-simplex numbers are polygonal. Note that 1-simplex numbers are line segments and 2-simplex numbers are triangular, so we skip past these trivialities and focus on $r \geq 3$ and $k \geq 3$. We will prove that almost all r-simplex numbers are polygonal, both asymptotically (Theorem 5) and experimentally (of the 198 smallest simplex numbers, only 4 fail to be polygonal).

We begin by recalling two key formulas, for the polygonal and simplex numbers respectively:

$$\mathcal{P} = \left\{ P_{s,n} = \frac{(s-2)n^2 - (s-4)n}{2} : s \geq 3, n \geq 3 \right\}, \text{ and}$$

$$S_r(k) = \binom{k+r-1}{r} = \frac{k(k+1)(k+2)\cdots(k+r-1)}{r!}.$$

[1]These are also known as polytopic numbers, sometimes denoted $P_r(k)$.

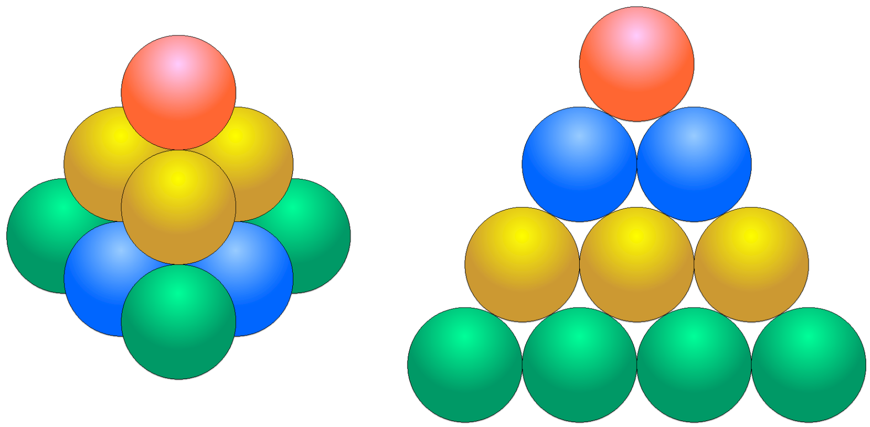

Figure 1: Examples of simplex and polygonal numbers

Left: Tetrahedral number $S_3(3) = 10$ (3 dimensional simplex, 3 per side) Right: Triangular number $P_{3,4} = 10$ (3-sided polygon, 4 per side)

We will need a theorem[2] proved by Édouard Lucas in a series of papers published in 1878 [3, 4, 5]. His other, later, work led to the well-known Lucas-Lehmer primality test. His life was tragically cut short at age 49 by a banquet accident[3]. More specifically, we will need this corollary to Lucas's Theorem[4].

Corollary to Lucas' Theorem. *Let p be prime and let $a, b \in \mathbb{N}$ with $a \geq b$. Express a, b in base p as $a = (a_j a_{j-1} \cdots a_1 a_0)_p, b = (b_j b_{j-1} \cdots b_1 b_0)_p$, where j is chosen so that $a_j \neq 0$. Then the following are equivalent:*

1. *p divides $\binom{a}{b}$;*
2. *There is some index $i \in [0, j]$ where $a_i < b_i$.*

It turns out that many r-simplex numbers are polygonal of the particular type $P_{s,3}$. These are characterized by our first theorem. Note that the characterization aligns well with the preceding corollary.

Theorem 1. *Let $r, k \in \mathbb{N}$ with $r, k \geq 3$. Then there is some $s \geq 3$ with $S_r(k) = P_{s,3}$ if and only if 3 divides $\binom{k+r-1}{r}$.*

[2] Curious French-speaking readers can find this in Section XXI of [4].

[3] A shard from a broken plate, dropped by a waiter, cut Lucas on the cheek. The infection that followed ended up killing him – pennicillin would not be discovered for decades.

[4] This states that, for any prime p, if we write $m, n \in \mathbb{N}$ in base p as $m = m_k \cdots m_1 m_0$ and $n = n_k \cdots n_1 n_0$, then $\binom{m}{n} = \prod_{i=0}^{k} \binom{m_i}{n_i} \pmod{p}$.

Proof. We begin by noting that

$$P_{s,3} = \frac{(s-2)3^2 - (s-4)3}{2} = 3s - 3.$$

Since we only allow $s \geq 3$, we see that $P_{s,3}$ takes on all multiples of 3 that are 6 or greater. Recall that $S_r(k) = \binom{k+r-1}{r}$. If $k \geq 3$, then $S_r(k) \geq \binom{r+2}{r} \geq 6$. Therefore it is of the type $P_{s,3}$ if and only if 3 divides $S_r(k)$. □

For example, Theorem 1 tells us that $P_3(7) = 84 = P_{29,3}$. Looking just at terminal ternary digits, we can produce infinitely many polygonal $S_r(k)$. Note that Theorem 2 is weaker than Theorem 1 – it does not tell us that $S_3(7)$ is polygonal, for example.

Theorem 2. *Let $r, k \in \mathbb{N}$ with $r, k \geq 3$. Then $S_r(k)$ is polygonal if*

$$(r \bmod 3, k \bmod 3) \in \{(2,0), (2,1), (1,0)\}.$$

Proof. We convert to base 3 as $r = (r_j \cdots r_1 r_0)_3$ and $k + r - 1 = (a_j \cdots a_1 a_0)_3$.

We now consider the case of $r \bmod 3 = 2$, i.e. $r_0 = 2$. Now, $k \not\equiv 1 \pmod 3$ and $r \equiv 2 \pmod 3$, so $k + r - 1 \not\equiv 2 \pmod 3$, so $a_0 < 2$. Applying the corollary to Lucas' Theorem, we find that $a_0 < r_0$, so 3 divides $\binom{k+r-1}{r}$.

We next consider the case of $r \bmod 3 = 1$, i.e. $r_0 = 1$. Since $k \equiv 0 \pmod 3$ and $r \equiv 1 \pmod 3$, we have $k + r - 1 \equiv 0 \pmod 3$, so $a_0 = 0$. Applying the corollary to Lucas' Theorem, we find that $a_0 < r_0$, so 3 again divides $\binom{k+r-1}{r}$.

Lastly, we apply Theorem 1 to find some $s \geq 3$ with $S_{r,k} = P_{s,3}$. □

Theorem 2 only looked at the terminal ternary digits. We can use similar methods to find many such theorems, but there are diminishing returns as we progress. We give one example, to illustrate why the returns diminish. It may seem at first glance that Theorem 3 covers $(3/9)(4/9) = 12/81 = 4/27$ of all remaining cases, but three of them[5] duplicate results from Theorem 2. Hence only $9/81 = 1/9$ of the remaining cases are covered by Theorem 3.

The smallest examples provided by Theorem 3 that are not covered by Theorem 2 are $S_6(4) = 84$ (which happens to equal $P_3(7)$, covered by Theorem 2) and $S_6(5) = 210$ (which isn't covered by Theorem 2, even indirectly).

Theorem 3. *Let $r, k \in \mathbb{N}$ with $r, k \geq 3$. Suppose that $r \bmod 9 \in \{6, 7, 8\}$, while $k \bmod 9 \in \{4, 5, 6, 7\}$. Then $S_r(k)$ is polygonal.*

Proof. We convert to base 3 as $r = (r_j \cdots r_1 r_0)_3$ and $k + r - 1 = (a_j \cdots a_1 a_0)_3$. Note that $r_1 = 2$ by hypothesis. We see that

$$(k \bmod 9) + (r \bmod 9) - 1 \in \{9, 10, \ldots, 14\},$$

so $k + r - 1 \bmod 9 \in \{0, 1, 2, 3, 4, 5\}$, and so $a_1 \in \{0, 1\}$. We now apply the corollary to Lucas' Theorem, and Theorem 1, as before. □

[5] $(r \bmod 9, k \bmod 9) \in \{(8,5), (8,6), (7,6)\}$

There are also analogues to Theorem 1 for other values of n. One such theorem follows. The smallest nontrivial example Theorem 4 provides is that $S_3(5) = 35 = P_{5,5}$.

Theorem 4. *Let $r, k \in \mathbb{N}$ with $r, k \geq 3$. Then there is some $s \geq 3$ with $S_r(k) = P_{s,5}$ if and only if 5 divides $\binom{k+r-1}{r}$ and 2 does not.*

Proof. We begin by noting that

$$P_{s,5} = \frac{(s-2)5^2 - (s-4)5}{2} = 10s - 15.$$

Since we only allow $s \geq 3$, we see that $P_{s,5}$ takes on all odd multiples of 5 that are greater than 5, i.e., $15, 25, 35, \ldots$. Recall that $S_r(k) = \binom{k+r-1}{r}$. If $k \geq 3$, then $S_r(k) \geq \binom{r+2}{r} \geq 6$. Therefore it is of the type $P_{s,3}$ if and only if $S_r(k) \equiv 5 \pmod{10}$. By the Chinese Remainder Theorem, this holds if and only if $S_r(k) \equiv 0 \pmod 5$ and $S_r(k) \equiv 1 \pmod 2$. ☐

We now provide an asymptotic result. By looking at all of the digits, we can prove that $S_r(k)$ is polygonal with high probability (just using $n = 3$) .

Theorem 5. *Let $N \in \mathbb{N}$, and let r, k each be chosen independently, uniformly at random, from $[0, 3^N)$. Then $S_r(k)$ is polygonal with probability at least*

$$(1 - (1/3)^{N-1})^2(1 - (2/3)^N) \geq 1 - 3(2/3)^N.$$

Proof. Each of r, k is at least 3 with probability $1 - 3^{N-1}$. Condition on this assumption. Now, each ternary digit of r, k is equally likely to be $0, 1, 2$, so each ternary digit of $k + r - 1, r$ are equally likely to be $0, 1, 2$. Of these nine cases, three – $(1, 2), (0, 1), (0, 2)$ – guarantee that $S_r(k)$ is polygonal via the corollary to Lucas' Theorem, as well as Theorem 1. To fail this test with all N digits is of probability $(2/3)^N$, so to pass it is of probability $1 - (2/3)^N$. Finally,

$$\begin{aligned}
&(1 - (1/3)^{N-1})^2(1 - (2/3)^N) = \\
&1 - (2/3)^N - 2(1/3)^{N-1} + (1/3)^{N-1}(2(2/3)^N + (1/3)^{N-1}(1 - (2/3)^N)) \geq \\
&\qquad 1 - (2/3)^N - 2(1/3)^{N-1} \geq 1 - (1 + 6/2^N)(2/3)^N \geq 1 - 3(2/3)^N,
\end{aligned}$$

where the final inequality holds for all $N \geq 2$ (and we check the trivial $N = 1$ separately). ☐

Our last two results are specifically for small and familiar values of r: $r = 3$ are the tetrahedral numbers and $r = 4$ are the pentatope numbers. The nicer result is for the latter.

Theorem 6. *Let $k \geq 3$. Then the pentatope number $S_4(k)$ is polygonal.*

Proof. If $k \equiv 0 \pmod 3$, then Theorem 2 shows that $S_4(k)$ is polygonal, with $n = 3$. Otherwise $S_4(k)$ will be polygonal with $s = 5$. We set $q = \frac{(k+1)(k+2)}{6}$. Note that q is an integer exactly when $k \not\equiv 0 \pmod 3$. We directly calculate

$$P_{5,q} = \frac{3q^2 - q}{2} = \frac{k^4}{24} + \frac{k^3}{4} + \frac{11k^2}{24} + \frac{k}{4} = \frac{k(k+1)(k+2)(k+3)}{24} = S_4(k).$$

□

Lastly, we consider tetrahedral numbers. Theorem 2 does not help here, unfortunately. However, we find that almost all tetrahedral numbers are polygonal. The cases of Theorem 7 may seem strange and random, but in fact there are often few choices of (n, s) that work, or only one! We challenge the reader to find a unifying pattern, and to resolve the missing case.

Theorem 7. *Let $k \geq 7$. If $k \not\equiv 6 \pmod{18}$, then $S_3(k)$ is polygonal.*

Proof. The proof proceeds in five cases. In each case, just as in Theorem 6, we explicitly compute $P_{s,n}$ and $S_3(k)$ to see they are equal.
Case $k \equiv 2 \pmod 3$: Take $n = k, s = \frac{k+10}{3}$.
Case $k \equiv 1 \pmod 3$: Take $n = \frac{k+2}{3}, s = 3k + 8$.
Case $k \equiv 0 \pmod 9$: Take $n = 3, s = 1 + \frac{k(k+1)(k+2)}{18}$.
Case $k \equiv \pm 3 \pmod{18}$: Take $n = \frac{k+1}{2}, s = 6 + \frac{4k}{3}$.
Case $k \equiv 12 \pmod{18}$: Take $n = 4, s = \frac{48+k(k+1)(k+2)}{36}$. □

Looking at the small values of k, we see that $S_3(3) = 10 = P_{3,3}$ (as in Figure 1) and $S_3(5) = 35 = P_{5,5}$; $S_3(4)$ and $S_3(6)$ are not polygonal. The "missing" tetrahedral case of $k \equiv 6 \pmod{18}$ really is not missing: many of those are computed to not be polygonal, in a complicated pattern.

In fact, these many positive results may give the impression that almost all simplex numbers are polygonal, but this is just due to a difficulty in proving negative results. We conjecture that there are infinitely many simplex numbers that fail to be polygonal. Even more strongly, we conjecture that there are infinitely many tetrahedral numbers that fail to be polygonal.

These conjectures are supported with some experimental evidence. The first five simplex numbers that fail to be polygonal are $20, 56, 2600, 6188$, and 13244. The quantity of simplex numbers that fail to be polygonal, up to various thresholds, are summarized in the table below.

$\leq 10^2$	$\leq 10^3$	$\leq 10^4$	$\leq 10^5$	$\leq 10^6$	$\leq 10^7$	$\leq 10^8$	$\leq 10^9$	$\leq 10^{10}$
2	2	4	9	16	24	40	75	151

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