Numerical Semigroups on Compound Sequences

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Abstract

We generalize the geometric sequence $\{a^p, a^{p-1}b, a^{p-2}b^2, \ldots, b^p\}$ to allow the p copies of a (resp. b) to all be different. We call the sequence $\{a_1a_2a_3\cdots a_p, b_1a_2a_3\cdots a_p, b_1b_2a_3\cdots a_p, \ldots, b_1b_2b_3\cdots b_p\}$ a *compound sequence*. We consider numerical semigroups whose minimal set of generators form a compound sequence, and compute various semigroup and arithmetical invariants, including the Frobenius number, Apéry sets, Betti elements, and catenary degree. We compute bounds on the delta set and the tame degree.

Keywords: nonunique factorization, numerical semigroup

Mathematics Subject Classification (2010): 20M13, 11A51, 20M14

1 Introduction

Let $\mathbb{N}$ denote the set of positive integers, and $\mathbb{N}_0$ denote the set of nonnegative integers. We call S a *numerical semigroup* if $S \subseteq \mathbb{N}_0$, S is closed under addition, S contains 0, and $|\mathbb{N} \setminus S| < \infty$. We say $\{x_0, x_1, \ldots, x_p\}$ is a *set of generators* for S if $S = \{\sum_{i=0}^{p} a_i x_i : a_i \in \mathbb{N}_0\}$, and call it minimal if it is minimal as ordered by inclusion. In this case we say S has *embedding dimension* $p+1$. For a general introduction to numerical semigroups, please see the monograph [18].

Here we will consider structure of numerical semigroups of a particular type, including some of its arithmetic properties. More generally, factorization theory studies the arithmetic properties of commutative, cancellative monoids and domains, where unique factorization fails to hold. For a general reference see any of [1, 2, 14].

If S is a numerical semigroup with minimal set of generators $\{n_0, n_1, \ldots, n_p\}$, the map

$$\phi : \mathbb{N}_0^{p+1} \to S, \quad \phi(x_0, x_1, \ldots, x_p) = x_0 n_0 + x_1 n_1 + \cdots + x_p n_p$$

is a monoid homomorphism, called the *factorization homomorphism* of S. Let σ be its kernel congruence, that is $x\sigma y$ if and only if $\phi(x) = \phi(y)$. Then S is isomorphic to $\mathbb{N}_0^{p+1}/\sigma$. We will consider σ as a subset of $\mathbb{N}_0^{p+1} \times \mathbb{N}_0^{p+1}$. Set $\mathcal{I}(S)$ to be the irreducibles of σ, viewed as a monoid.

For $n \in S$, the set $\phi^{-1}(n)$ is the set of *factorizations* of n. We say $n > 1$ is a *Betti element* if there is a partition $\phi^{-1}(n) = X \cup Y$ satisfying $\sum_{i=0}^{p} x_i y_i = 0$ for each $x \in X, y \in Y$. Betti elements capture important semigroup structure, and have received considerable recent attention ([6, 10, 11, 12]). If $x = (x_0, \ldots, x_p) \in \phi^{-1}(n)$, the *length* of the factorization x is $|x| = x_0 + \cdots + x_p$. If $x, y \in \mathbb{N}_0^{p+1}$, we define

$$\gcd(x, y) = (\min\{x_0, y_0\}, \min\{x_1, y_1\}, \ldots \min\{x_p, y_p\}) \in \mathbb{N}_0^{p+1}.$$

We also define the distance between x and y as

$$d(x, y) = \max\{|x - \gcd(x, y)|, |y - \gcd(x, y)|\}.$$

Further, for $Y \subseteq \mathbb{N}_0^{p+1}$, we define $d(x, Y) = \min\{d(x, y) : y \in Y\}$. Given $n \in S$ and $x, y \in \phi^{-1}(n)$, then a *chain of factorizations* from x to y is a sequence

$x^0, x^1, \dots x^k \in \phi^{-1}(n)$ such that $x^0 = x$ and $x^k = y$. We call this an N-*chain* if $d(x^i, x^{i+1}) \leq N$ for all $i \in [0, k-1]$. The *catenary degree* of n, $c(n)$, is the minimal $N \in \mathbb{N}_0$ such that for any two factorizations $x, y \in \phi^{-1}(n)$, there is an N-chain from x to y. The catenary degree of S, $c(S)$, is defined by

$$c(S) = \sup\{c(n) : n \in S\}.$$

For a semigroup S and $n \in S$, we define the *length set* of n as $\mathcal{L}(n) = \{|x| : x \in \phi^{-1}(n)\}$. If we label $\mathcal{L}(n) = \{t_1, t_2, \dots, t_k\}$ with $t_1 < t_2 < \cdots < t_k$, then we define the *delta set* of n as $\Delta(n) = \{t_i - t_{i-1} : i \in [2, k]\}$, with $\Delta(n) = \emptyset$ if $|\mathcal{L}(n)| = 1$. We define the *delta set* of S as $\Delta(S) = \cup_{n \in S} \Delta(n)$. For $i \in [0, p]$, and $n \in S$, we define $\phi_i^{-1}(n) = \{(x_0, \dots, x_p) \in \phi^{-1}(n) : x_i > 0\}$. We define $t_i(n) = \max\{d(z, \phi_i^{-1}(n)) : z \in \phi^{-1}(n)\}$ for $\phi_i^{-1}(n) \neq \emptyset$, and set $t_i(n) = -\infty$ otherwise. We define the *tame degree* of n as $t(n) = \max\{t_i(n) : i \in [0, p]\}$, and the *tame degree* of S as $t(S) = \max\{t(n) : n \in S\}$. For more background on arithmetic invariants in general numerical semigroups see [5, 7].

Numerical semigroups whose minimal generators are geometric sequences $\langle a^p, a^{p-1}b, a^{p-2}b^2, \dots, b^p \rangle$ have been investigated recently in [17, 20]. We propose a generalization of such sequences, which we call *compound sequences.*

Definition 1. *Let $p, a_1, a_2, \dots, a_p, b_1, b_2, \dots, b_p \in \mathbb{N}$. Suppose that:*

1. $2 \leq a_i < b_i$*, for each* $i \in [1, p]$.
2. $\gcd(a_i, b_j) = 1$ *for all* $i, j \in [1, p]$ *with* $i \geq j$.

For each $i \in [0, p]$, we set $n_i = b_1 b_2 \cdots b_i a_{i+1} a_{i+2} \cdots a_p$. We then call the sequence $\{n_0, n_1, \dots, n_p\}$ a compound sequence.

Applying this definition repeatedly leads to the particular consequences $\gcd(a_i, b_i) = 1, \gcd(a_i, b_1 b_2 \cdots b_i) = 1$, $\gcd(a_i a_{i+1} \cdots a_p, b_i) = 1$, and lastly

$\gcd(a_i a_{i+1}\cdots a_p, b_1 b_2\cdots b_i) = 1$. Note that the special case of $a_1 = a_2 = \cdots = a_p$, $b_1 = b_2 = \cdots = b_p$ gives a geometric sequence. We now present some elementary properties of compound sequences.

Proposition 2. *Let $\{n_0, n_1, \ldots, n_p\}$ be a compound sequence as defined above. Then the following all hold.*

1. $n_i = \frac{b_i}{a_i} n_{i-1}$, *for each* $i \in [1, p]$.
2. $n_0 < n_1 < \ldots < n_p$.
3. $\gcd(n_0, n_1, \ldots, n_i) = \prod_{j=i+1}^{p} a_j$ *for all* $i \in [0, p]$.
4. $\gcd(n_i, n_{i+1}, \ldots, n_p) = \prod_{j=1}^{i} b_j$ *for all* $i \in [0, p]$.
5. $\gcd(n_0, n_1, \ldots, n_p) = 1$.
6. $\langle n_0, n_1, \ldots, n_p\rangle$ *is a minimally generated numerical semigroup.*
7. $a_i = \frac{n_{i-1}}{\gcd(n_{i-1}, n_i)}$ *and* $b_i = \frac{n_i}{\gcd(n_{i-1}, n_i)}$, *for each* $i \in [1, p]$.

Proof. (1) is immediate from the definition.
(2) follows from (1) since $\frac{n_i}{n_{i-1}} = \frac{b_i}{a_i} > 1$ for each $i \in [1, p]$.
(3) Set $A = \prod_{j=i+1}^{p} a_j$. Since A divides each of $n_0, \ldots, n_i$, it suffices to prove that $\gcd(n'_0, \ldots, n'_i) = 1$, where $n'_0 = \frac{n_0}{A}, \ldots, n'_i = \frac{n_i}{A}$. Suppose prime q divides $\gcd(n'_0, \ldots, n'_i)$. Then $q | \gcd(n'_0, n'_i) = \gcd(a_1 a_2 \cdots a_i, b_1 b_2 \cdots b_i)$. Let k be maximal in $[1, i]$ so that $q | a_k$, and let j be minimal in $[1, i]$ so that $q | b_j$. Since $q | \gcd(a_k, b_j)$, by the definition of compound sequences we must have $k < j$. But now $q \nmid n'_k$, a contradiction.
(4) Set $B = \prod_{j=1}^{i} b_j$. Since B divides each of $n_i, \ldots, n_p$, it suffices to prove that $\gcd(n'_i, \ldots, n'_p) = 1$, where $n'_i = \frac{n_i}{B}, \ldots, n'_p = \frac{n_p}{B}$. Suppose prime q divides $\gcd(n'_i, \ldots, n'_p)$. Then $q | \gcd(n'_i, n'_p) = \gcd(a_{i+1} \cdots a_p, b_{i+1} \cdots b_p)$. Let k be maximal in $[i+1, p]$ so that $q | a_k$, and let j be minimal in $[i+1, p]$ so that

$q|b_j$. Since $q|\gcd(a_k, b_j)$, by the definition of compound sequences we must have $k < j$. But now $q \nmid n'_k$, a contradiction.

(5) Follows from either (3) or (4).

(6) This is a numerical semigroup by (5). To prove it is minimally generated, we appeal to Cor. 1.9 from [18], by which it suffices to prove that $n_i \notin \langle n_0, \dots, n_{i-1} \rangle$ for each $i \in [1, p]$. Set $x = a_i a_{i+1} \cdots a_p$. We have $x|\gcd(n_0, n_1, \dots, n_{i-1})$. If $n_i \in \langle n_0, \dots, n_{i-1} \rangle$ then $x|n_i = b_1 b_2 \cdots b_i a_{i+1} a_{i+2} \cdots a_p$. Cancelling, we get $a_i|b_1 b_2 \cdots b_i$, a contradiction since $a_i > 1$ yet $\gcd(a_i, b_1 b_2 \cdots b_i) = 1$.

(7) follows by combining $\gcd(n_{i-1}, n_i) = b_1 b_2 \cdots b_{i-1} a_{i+1} a_{i+2} \cdots a_p \gcd(a_i, b_i)$ with $\gcd(a_i, b_i) = 1$. □

Note that Proposition 2.7 suggests that the generators $n_0, \dots n_p$ alone suffice to recover the $\{a_i\}, \{b_i\}$. This is indeed the case, as shown in the following.

Proposition 3. *Let $n_0, n_1, \dots, n_p \in \mathbb{N}$ with $n_0 < n_1 < \cdots < n_p$. Suppose that $\langle n_0, n_1, \dots, n_p \rangle$ is a minimally generated numerical semigroup. Then the following are equivalent.*

1. *$\{n_0, n_1, \dots, n_p\}$ is a compound sequence.*
2. *$n_1 n_2 \cdots n_{p-1} = \gcd(n_0, n_1) \gcd(n_1, n_2) \cdots \gcd(n_{p-1}, n_p)$*

Proof. ($1 \to 2$). Applying Prop. 2.7, we have $\frac{n_1}{\gcd(n_1,n_2)} \frac{n_2}{\gcd(n_2,n_3)} \cdots \frac{n_{p-1}}{\gcd(n_{p-1},n_p)} = a_2 a_3 \cdots a_p = \gcd(n_0, n_1)$. Cross-multiplying yields (2).

($2 \to 1$). We define a_i, b_i as in Proposition 2.7. Note that $a_i n_i = b_i n_{i-1}$ and that $\gcd(a_i, b_i) = 1$. Also note that $a_i < b_i$ since $n_{i-1} < n_i$, and that $a_i > 1$ since otherwise $n_{i-1}|n_i$ but the semigroup is minimally generated. Dividing both sides of (2) by $\gcd(n_1, n_2) \cdots \gcd(n_{p-1}, n_p)$, we get $a_2 a_3 \cdots a_p = \gcd(n_0, n_1)$. Since $a_1 = \frac{n_0}{\gcd(n_0,n_1)}$ we conclude that $n_0 = a_1 a_2 \cdots a_p$. Repeatedly applying $a_i n_i = b_i n_{i-1}$ gives $n_i = b_1 b_2 \cdots b_i a_{i+1} a_{i+2} \cdots a_p$ for $i \in [0, p]$. Lastly, if $\gcd(a_i, b_1 b_2 \cdots b_i) = d > 1$ for some i, then d divides each of $n_0, n_1, \dots, n_p$. This contradicts $\langle n_0, n_1, \ldots, n_p \rangle$ being a numerical semigroup since $\gcd(n_0, \ldots, n_p) \geq d$. $\square$

Applying Proposition 3, we see that in embedding dimension 2, every numerical semigroup $\langle a, b \rangle$ is on a compound sequence. Further, in embedding dimension 3, we see that numerical semigroup $\langle a, b, c \rangle$ is on a compound sequence if and only if we can write $b = b_1 b_2$ where $b_1 | a$ and $b_2 | c$. Henceforth we will focus on numerical semigroups on compound sequences, which we will abbreviate as NSCS.

Such semigroups are not too rare. For example, consider numerical semigroups of embedding dimension 3, whose largest generator is at most 200. Of these, 1% have their generators in a compound sequence, while 0.6% have their generators in an arithmetic sequence. The latter class of semigroups, and variations thereof, has been the subject of much recent study in [3, 5, 8, 15, 16].

2 Factorization Structure

We now turn to the study of factorizations in an NSCS. These have very nice structure, which will be developed in this section. For nonzero $x \in \mathbb{Z}^{p+1}$, we define $\min(x) = \min\{i : x_i \neq 0\}$ and $\max(x) = \max\{i : x_i \neq 0\}$. Note that for any $x, y \in \mathbb{N}_0^{p+1}$, $\min(x) \geq \min(x+y)$ and $\max(x) \leq \max(x+y)$. Note also that $\min(x-y)$ is the smallest coordinate where x, y differ. This next, technical, result divides factorizations of the important element $a_i n_i = b_i n_{i-1}$ into two quite different categories. In particular, it implies that they are each Betti elements.

Proposition 4. *Let $S = \langle n_0, \ldots, n_p \rangle$ be an NSCS, and let $i \in [1, p]$. Let $x \in \phi^{-1}(a_i n_i)$. Then one of the following must hold:*

1. $\min(x) \geq i$ *and* $|x| \leq a_i$*; or*

2. $\max(x) \le i-1$ *and* $|x| \ge b_i$.

Further, factorizations of both types exist, where all inequalities are met.

Proof. Set $a = a_i a_{i+1} \cdots a_p, b = b_1 b_2 \cdots b_i$. Note that $a_i n_i = ab$ and that a divides each of $n_0, n_1, \ldots, n_{i-1}$ while b divides each of $n_i, n_{i+1}, \ldots, n_p$. We calculate modulo b: $0 \equiv a_i n_i \equiv \sum_{j=0}^{p} x_j n_j \equiv \sum_{j=0}^{i-1} x_j n_j$. We divide both sides by a (since $\gcd(a,b) = 1$) to get $0 \equiv \sum_{j=0}^{i-1} x_j \frac{n_j}{a} \pmod{b}$. If $\sum_{j=0}^{i-1} x_j \frac{n_j}{a} = 0$, then $\min(x) \ge i$. Otherwise, $b \le \sum_{j=0}^{i-1} x_j \frac{n_j}{a}$ and we multiply both sides by a to get $ab \le \sum_{j=0}^{i-1} x_j n_j \le \sum_{j=0}^{p} x_j n_j = a_i n_i = ab$. All the inequalities are equalities and hence $\max(x) \le i-1$.

Now, partition $\phi^{-1}(n) = X \cup Y$, where factorizations $x \in X$ satisfy $\min(x) \ge i$ and factorizations $y \in Y$ satisfy $\max(y) \le i-1$. For any $x \in X$, we have $n = a_i n_i = \sum_{j=i}^{p} x_j n_j \ge |x| n_i$, and hence $|x| \le a_i$. Similarly, for any $y \in Y$, we have $n = b_i n_{i-1} = \sum_{j=1}^{i-1} y_j n_j \le |y| n_{i-1}$, and hence $|y| \ge b_i$.

Finally, note that $a_i e_i \in X$ and $b_i e_{i-1} \in Y$. $\square$

This next lemma is essential for the proof of Theorem 8, and relates two factorizations of the same element, on their extremal coordinates.

Lemma 5. *Let* $S = \langle n_0, \ldots, n_p \rangle$ *be an NSCS. Let* $n \in S$*, and* $x, y \in \phi^{-1}(n)$*. Set* $m = \min(x+y), m' = \max(x+y)$*. Then* $x_m \equiv y_m \pmod{b_{m+1}}$ *and* $x_{m'} \equiv y_{m'} \pmod{a_{m'}}$.

Proof. Set $b = b_1 b_2 \cdots b_{m+1}$. Note that b divides each of $n_{m+1}, n_{m+2}, \ldots, n_p$. Hence $n \pmod{b} \equiv x_m n_m + x_{m+1} n_{m+1} + \cdots x_p n_p \pmod{b} \equiv x_m n_m \pmod{b}$ but equally $n \pmod{b} \equiv y_m n_m \pmod{b}$. We conclude that $b | (x_m - y_m) n_m$, or $(b_1 b_2 \cdots b_{m+1}) | (x_m - y_m)(b_1 b_2 \cdots b_m a_{m+1} \cdots a_p)$. Cancelling, we get $b_{m+1} | (x_m - y_m)(a_{m+1} \cdots a_p)$. Hence $b_{m+1} | (x_m - y_m)$ as $\gcd(a_{m+1} \cdots a_p, b_{m+1}) = 1$.

Now set $a = a_{m'} a_{m'+1} \cdots a_p$. Note that a divides each of $n_0, n_1, \ldots, n_{m'-1}$. Hence $n \pmod{a} \equiv x_0 n_0 + \cdots + x_{m'-1} n_{m'-1} + x_{m'} n_{m'} \pmod{a} \equiv x_{m'} n_{m'}$

(mod a) but equally $n \pmod{a} \equiv y_{m'}n_{m'} \pmod{a}$. We conclude that $a|(x_{m'} - y_{m'})n_{m'}$, or $(a_{m'}a_{m'+1}\cdots a_p)|(x_{m'} - y_{m'})(b_1b_2\cdots b_{m'}a_{m'+1}\cdots a_p)$. Cancelling, we get $a_{m'}|(x_{m'}-y_{m'})(b_1\cdots b_{m'})$. Hence $a_{m'}|(x_{m'}-y_{m'})$ as $\gcd(a_{m'}, b_1\cdots b_{m'}) = 1$. □

Definition 6. *For a fixed NSCS $\langle n_0, \ldots, n_p\rangle$, we now define* basic swaps*. These are elements of the kernel congruence σ, for each $i \in [1, p]$, given by*

$$\delta_i = (a_ie_i, b_ie_{i-1}), \quad \delta_i' = (b_ie_{i-1}, a_ie_i)$$

We define $\Omega = \{\delta_i\} \cup \{\delta_i'\}$, the set of all basic swaps. For $\tau = (\tau_1, \tau_2) \in \Omega$, if $x + \tau_1 = y + \tau_2$, we say that we apply the basic swap τ to get from x to y*. If $x^0, x^1, \ldots, x^k$ is a chain of factorizations in $\phi^{-1}(n)$, we call this a* basic chain *if for each $i \in [1, k-1]$ we get from x^{i+1} to x^i by applying $\tau_i \in \Omega$. If a basic chain also satisfies, for all $i \in [1, k-1]$, that $\tau_i \in \{\delta_j, \delta_j'\}$, where $j = 1+\min(x_{i-1}-x_i)$, we call it a* left-first basic chain*. Similarly, if a basic chain also satisfies, for all $i \in [1, k-1]$, that $\tau_i \in \{\delta_j, \delta_j'\}$, where $j = \max(x_{i-1} - x_i)$, we call it a* right-first basic chain*.*

Note that if z is part of either a left-first or right-first basic chain from x to y, then $\min(z) \geq \min(x+y)$ and $\max(z) \leq \max(x+y)$. Note also that if we apply basic swap δ_i (or δ_i') to get from x to y, then $d(x, y) = d(a_ie_i, b_ie_{i-1}) = b_i$. Each basic swap is in σ since $a_in_i = b_in_{i-1}$, but in fact basic swaps are irreducibles in σ, as shown by the following.

Lemma 7. *Let $S = \langle n_0, \ldots, n_p\rangle$ be an NSCS. Then $\Omega \subseteq \mathcal{I}(\sigma)$.*

Proof. We consider fixed δ_i (the case of δ_i' is similar). If it were reducible, then there is some $(\alpha e_i, \beta e_{i-1}) \in \sigma$, with $0 < \alpha < a_i$. Hence $\alpha b_1 \cdots b_{i-1}b_ia_{i+1}\cdots a_P = \phi(\alpha e_i) = \phi(\beta e_{i-1}) = \beta b_1 \cdots b_{i-1}a_ia_{i+1}\cdots a_P$. Cancelling, we get $\alpha b_i = \beta a_i$ and hence $\alpha b_i \equiv 0 \pmod{a_i}$. Since $\gcd(a_i, b_i) = 1$, in fact $\alpha \equiv 0 \pmod{a_i}$, a contradiction. □

The following theorem proves the existence of basic chains connecting any two factorizations. Combined with Lemma 7, it implies that Ω is a minimal presentation of σ.

Theorem 8. *Let $S = \langle n_0, \ldots, n_p \rangle$ be an NSCS. Let $n \in S$, and $x, y \in \phi^{-1}(n)$. Then there are both left-first and right-first basic chains of factorizations from x to y.*

Proof. We will only prove the existence of a left-first basic chain (the right-first case is similar). We argue by way of contradiction. Let n be minimal possessing at least one pair of factorizations $x, y \in \phi^{-1}(n)$ that do not admit a left-first basic chain between them. Of all such pairs in $\phi^{-1}(n)$ not admitting a basic chain, choose a pair $x, y \in \phi^{-1}(n)$ with $|x_{\min(x+y)} - y_{\min(x+y)}|$ minimal. For convenience, set $t = \min(x+y)$. Depending on whether $x_t - y_t$ is positive, negative, or zero, we now have three cases, each of which will lead to contradiction.

Suppose first that $x_t - y_t > 0$. By Lemma 5, in fact $x_t \geq y_t + b_{t+1}$. We now apply δ_{t+1} to x to get $z = x - b_{t+1}e_t + a_{t+1}e_{t+1}$. Since $z \in \phi^{-1}(n)$ and $|z_t - y_t| < |x_t - y_t|$, there must be a left-first basic chain of factorizations $z^0, z^1, \ldots, z^k$ from z to y. But then $x, z^0, z^1, \ldots, z^k$ is a left-first basic chain of factorizations from x to y, which is a contradiction.

Suppose next that $x_t - y_t < 0$. By Lemma 5, in fact $y_t \geq x_t + b_{t+1}$. We now apply δ_{t+1} to y to get $z = y - b_{t+1}e_t + a_{t+1}e_{t+1}$. Since $z \in \phi^{-1}(n)$ and $|x_t - z_t| < |x_t - y_t|$, there must be a left-first basic chain of factorizations $x^0, x^1, \ldots, x^k$ from x to z. But then $x^0, x^1, \ldots, x^k, y$ is a left-first basic chain of factorizations from x to y, which is a contradiction.

Lastly we have $x_t = y_t$, with $x_t > 0$. We now set $\bar{n} = n - x_t n_t$, $\bar{x} = x - x_t e_t$, $\bar{y} = y - y_t e_t$. Since $\bar{n} < n$, by the choice of n any two factorizations of $\bar{n}$ must admit a left-first basic chain between them. In particular, $\bar{x}, \bar{y} \in \phi^{-1}(\bar{n})$ must admit a left-first basic chain $\bar{x}^0, \bar{x}^1, \ldots, \bar{x}^k$. But then $(\bar{x}^0 + x_t e_t), (\bar{x}^1 + x_t e_t), \ldots, (\bar{x}^k + x_t e_t)$ is a left-first basic chain from x to y, which is a contradiction. □

We recall that a numerical semigroup is a complete intersection if the cardinality each of its minimal presentations is one less than its embedding dimension. We recall that a numerical semigroup is free if for some ordering of its generators $n'_1, \ldots, n'_p$, and for all $i \in [2,p]$, we have $\min\{k \in \mathbb{N} : kn'_i \in \langle n'_1, \ldots, n'_{i-1}\rangle\} = \min\{k \in \mathbb{N} : kn'_i \in \langle n'_1, \ldots, n'_{i-1}, n'_{i+1}, \ldots, n'_p\rangle\}$.

Corollary 9. *Let $S = \langle n_0, \ldots, n_p\rangle$ be an NSCS. Then S is a free numerical semigroup, and a complete intersection.*

Proof. Corollaries 8.17 and 8.19 of [18]. □

Corollary 10. *Let $S = \langle n_0, \ldots, n_p\rangle$ be an NSCS. Then $\{a_1 n_1, a_2 n_2, \ldots, a_p n_p\}$ is the set of Betti elements of S.*

Proof. Let $n \in S$ be a Betti element. Hence there is a partition $\phi^{-1}(n) = X \cup Y$, where $\sum_{i=0}^p x_i y_i = 0$ for each $x \in X, y \in Y$. Choose $x \in X, y \in Y$. Take a basic chain of factorizations from x to y. There must be some consecutive factorizations x^k, x^{k+1} in this chain, where $x^k \in X$ and $x^{k+1} \in Y$. Hence for some $j \in [1,p]$, we have $x^{k+1} = x^k - a_j e_j + b_j e_{j-1}$ (or, similarly, $x^{k+1} = x^k + a_j e_j - b_j e_{j-1}$). We have $0 = x^k \cdot x^{k+1}$, but $x^k, x^{k+1} \in \mathbb{N}_0^{p+1}$, so $x^k = a_j e_j$ and hence $\phi(x^k) = a_j n_j$. Proposition 4 provides the other direction. □

We now define i-normal factorizations in an NSCS, which will be of use later.

Definition 11. *For a fixed NSCS $S = \langle n_0, \ldots, n_p\rangle$, a fixed $n \in S$, and a fixed $i \in [0,p]$, we call factorization $x \in \phi^{-1}(n)$* i-normal *if it satisfies:*

1. *for all $j < i$, $0 \le x_j < b_{j+1}$; and*

2. for all $j > i$, $0 \leq x_j < a_j$.

Note that these conditions are equivalent to none of the basic swaps in the set $\{\delta_1, \delta_2, \ldots, \delta_i, \delta'_{i+1}, \delta'_{i+2}, \ldots, \delta'_p\}$ applying to x. The following proposition justifies calling the term "normal".

Proposition 12. *Let $S = \langle n_0, \ldots, n_p \rangle$ be an NSCS. Let $n \in S$, and let $i \in [0, p]$. Then there is exactly one $x \in \phi^{-1}(n)$ that is i-normal.*

Proof. For a factorization, call a coordinate "good" if it satisfies the appropriate condition of i-normality, and "bad" otherwise (coordinate i is neither). We will prove existence of i-normal factorizations, in two stages. First, we prove that there are factorizations that satisfy the first i-normal condition. If not, choose x so that its smallest bad coordinate, s, is maximal. That is, $0 \leq x_j < b_{j+1}$ for $j \in [0, s-1]$, but $x_s \geq b_{s+1}$. Set $t = \lfloor \frac{x_s}{b_{s+1}} \rfloor \geq 1$, and set $y = x + t\delta_{s+1}$. By construction of t, we have $0 \leq y_s < b_{s+1}$. Hence the smallest "bad" coordinate of y is greater than the smallest "bad" coordinate of x, a contradiction.

Next, we consider only factorizations that satisfy the first i-normal condition; these exist by the previous. We will prove (at least) one of these satisfies the second i-normal condition. If not, choose x so that its largest bad coordinate, s, is minimal. That is, $0 \leq x_j < a_j$ for $j \in [s+1, p]$, but $x_s \geq a_s$. Set $t = \lfloor \frac{x_s}{a_s} \rfloor \geq 1$, and set $y = x - t\delta_s$. By construction of t, we have $0 \leq y_s < a_s$. Hence the largest "bad" coordinate of y is smaller than the largest "bad" coordinate of x, a contradiction.

We now prove uniqueness. Let x, y be i-normal factorizations of n. Set $s = \min(x - y)$. Suppose that $s < i$. We set $z = (x_0, x_1, \ldots, x_{s-1}, 0, 0, \ldots, 0)$, and apply Lemma 5 to $x - z, y - z$, both factorizations of $n - \phi(z)$. We conclude that $x_s \equiv y_s \pmod{b_{s+1}}$; however since x, y are simple in fact $x_s = y_s$, a contradiction. Hence $\min(x - y) \geq i$. By using the second i-normal condition, and the second part of Lemma 5, we similarly prove that $\max(x - y) \leq i$. Hence

x, y agree, except possibly for x_i, y_i. However if $x_i \neq y_i$ they would not be factorizations of the same n. □

This normal factorization yields various consequences, developed below. Our first observation is that i-normal factorizations are maximal in the i-th coordinate.

Corollary 13. *Let $S = \langle n_0, \ldots, n_p \rangle$ be an NSCS. Let $n \in S$, and let $i \in [0, p]$. Let $x, y \in \phi^{-1}(s)$, and suppose that x is i-normal. Then $x_i \geq y_i$.*

Proof. Suppose that $y_i > x_i$. Then $n - \phi(y_i n_i) \in S$, and has an i-normal factorization z. But now $z + y_i e_i$ is an i-normal factorization for n, which contradicts the uniqueness of x. □

Note that since $a_i < b_i$, applying any basic swap δ_i decreases the factorization length, while applying any δ_i' increases the factorization length. This observation, together with Theorem 8 and the comments preceding Proposition 12, yield the following.

Corollary 14. *Let $S = \langle n_0, \ldots, n_p \rangle$ be an NSCS. Let $n \in S$. Then the minimum factorization length of n is the length of the p-normal factorization of n. Also, the maximum factorization length of n is the length of the 0-normal factorization of n.*

3 Apéry sets

For a semigroup S and $m \in S$, recall that an Apéry set is defined as

$$Ap(S, m) = \{n \in S : n - m \notin S\}.$$

These are most commonly computed when m is an irreducible; for this application i-normal forms prove to be very helpful. For $n \in S$, we let x be the

i-normal factorization for n. The following theorem proves that $n \in Ap(S, n_i)$ if and only if $x_i = 0$.

Theorem 15. *Let $S = \langle n_0, \ldots, n_p \rangle$ be an NSCS. Let $i \in [0, p]$. Then the Apéry set $Ap(S, n_i) = \{\phi(u) : u \in S_i\}$, where*

$$S_i = \left\{ u \in \mathbb{N}_0^{p+1} : \begin{array}{c} u_0 < b_1, u_1 < b_2, \ldots, u_{i-1} < b_i, u_i = 0, \\ u_{i+1} < a_{i+1}, u_{i+2} < a_{i+2}, \ldots, u_p < a_p \end{array} \right\}.$$

Proof. If $x \in S_i$, then x is i-normal, and hence by Corollary 13, $x_i = 0$ is maximal over all factorizations of $\phi(x)$. Hence $\phi(x - x_i) \notin S$, and $\phi(x) \in Ap(S, n_i)$. On the other hand, for $n \in Ap(S, n_i)$, let x be the i-normal factorization of n. If $x_i > 0$ then $n - n_i \in S$, which is impossible. Hence $x_i = 0$ and thus $x \in S_i$. □

For a numerical semigroup S, recall that the largest integer in $\mathbb{N} \setminus S$ is called the *Frobenius number* of S, denoted $g(S)$. In [4], Brauer and Shockley proved that $g(S) = \max Ap(S; m) - m$. Applying this to Theorem 15, with $i = 0$ for simplicity, we get the following corollary, which directly generalizes the main result of [17].

Corollary 16. *Let $S = \langle n_0, \ldots, n_p \rangle$ be an NSCS. Let $i \in [0, p]$. Then*

$$g(S) = -n_0 + \sum_{j=1}^{p} n_j (a_j - 1)$$

For a numerical semigroup S, recall that $|\mathbb{N} \setminus S|$ is called the *genus of* S, denoted $N(S)$. In [19], Selmer proved that $N(S) = -\frac{m-1}{2} + \frac{1}{m} \sum_{n \in Ap(S,m)} n$. Applying this to Theorem 15, with $i = 0$ for simplicity, we get the following corollary.

Corollary 17. *Let $S = \langle n_0, \ldots, n_p \rangle$ be an NSCS. Then*

$$N(S) = \frac{1}{2} \left(1 - n_0 + \sum_{j=1}^{p} n_j (a_j - 1) \right)$$

Proof. We compute $X = \sum_{u \in S_0} \phi(u)$. We write $X = X_1 + \cdots + X_P$, where $X_j = \sum_{u \in S_0} u_j n_j$. In the terms of X_j, u_j assumes each of the values $0, 1, \ldots, a_j - 1$ equally often, specifically $\frac{a_1 a_2 \cdots a_p}{a_j}$ times. Hence $X_j = (0 + 1 + \cdots + (a_j - 1))\frac{a_1 a_2 \cdots a_p}{a_j} n_j = \frac{a_j - 1}{2}(a_1 a_2 \cdots a_p) n_j$. Summing, we get $X = \frac{a_1 a_2 \cdots a_p}{2} \sum_{j=1}^{p} n_j (a_j - 1)$, and the result follows. □

Combining the previous two corollaries, we see that $N(S) = \frac{1+g(S)}{2}$, which is precisely the definition of *symmetric* numerical semigroups. Hence all NSCS semigroups are symmetric; this result also follows since they are complete intersections (see Cor. 8.12 in [18]).

4 Arithmetic Invariants

We now compute several arithmetic invariants in the NSCS context. First we consider the catenary degree $c(S)$, which we can determine exactly. In the special case of a geometric sequence $S = \langle a^p, a^{p-1}b, \ldots, b^p \rangle$, this gives $c(S) = b$.

Theorem 18. *Let $S = \langle n_0, \ldots, n_p \rangle$ be an NSCS. Then $c(S) = \max\{b_1, b_2, \ldots, b_p\}$.*

Proof. By Theorem 8, we may connect any two factorization by a basic chain. Hence $c(S) \leq \max\{b_1, \ldots, b_p\}$. Now fix i such that $b_i = \max\{b_1, b_2, \ldots, b_p\}$. Let x, y be two factorizations of $a_i n_i$ of the two types guaranteed by Proposition 4. We have $\gcd(x, y) = 0$ so $d(x, y) = \max\{|x|, |y|\} \geq b_i$. Any chain of factorizations connecting $a_i n_i$ to $b_i n_{i-1}$ must at some point cross from one factorization type to the other, a step of size at least b_i. Hence $c(a_i n_i) \geq b_i$, so $c(S) \geq \max\{b_1, \ldots, b_p\}$. □

In particular, Theorem 18 shows that the catenary degree of an NSCS is achieved at a Betti element. Compare this to the result in [11] that the catenary degree of a half-factorial numerical semigroup is achieved by a Betti element.

We now consider $\Delta(S)$ in our context, which we can partially determine. Recall from [13] that $\min(\Delta(S)) = \gcd(\Delta(S))$.

Theorem 19. *Let* $S = \langle n_0, \ldots, n_p \rangle$ *be an NSCS. Set* $N = \{b_1 - a_1, b_2 - a_2, \ldots, b_p - a_p\}$. *Then:*

1. $\min(\Delta(S)) = \gcd(N)$,
2. $N \subseteq \Delta(S)$, *and*
3. $\max(\Delta(S)) = \max(N)$.

Proof. (1) Each basic swap is irreducible in σ. Hence by Proposition 2.2 of [3], $\min(\Delta(S)) = \gcd(P(\sigma)) \leq \gcd(N)$. For the reverse direction, note that $n_i - n_{i-1} = (b_i - a_i)b_1 \cdots b_{i-1}a_{i+1} \cdots a_p$, so $\gcd(N) | \gcd(\{n_i - n_{i-1} : i \in [1,p]\})$, which equals $\min(\Delta(S)$ by Proposition 2.10 of [3].
(2) By Proposition 4, we see that $\mathcal{L}(a_i n_i)$ contains a_i, b_i, and no values in between. Hence $b_i - a_i \in \Delta(a_i n_i) \subseteq \Delta(S)$.
(3) Let $d \in \Delta(S)$. Then there is some $n \in S$ and $x, y \in \phi^{-1}(n)$ with $|y| = |x| + d$, and $\mathcal{L}(n)$ contains no integer strictly between $|x|$ and $|y|$. Applying Theorem 8 there is a basic chain of factorizations $x^0, x^1, \ldots, x^k$ from x to y. Hence $\{|x^0|, |x^1|, \ldots, |x^k|\} \subseteq \mathcal{L}(n)$. Note that $\big||x^{i-1}| - |x^i|\big| \in N$, for each $i \in [1,k]$, so $\mathcal{L}(x)$ contains a sequence of integers from $|x|$ to $|y|$, each at most $\max(N)$ away from the last. Thus $d \leq \max(N)$. This proves that $\max(\Delta(S)) \leq \max(N)$; combined with (2) the result follows. □

In certain cases, as shown below, Theorem 19 determines $\Delta(S)$ completely. In particular, the geometric sequence case is settled since that restriction implies $|N| = 1$.

Corollary 20. *Let* $S = \langle n_0, \ldots, n_p \rangle$ *be an NSCS. Set* $N = \{b_1 - a_1, b_2 - a_2, \ldots, b_n - a_n\}$. *Suppose that any of the following hold:*

1. $|N| = 1$*, or*

2. $|N| > 1$ *and for some* $\alpha \in \mathbb{N}$*,* $N = \{\alpha, 2\alpha, \ldots, |N|\alpha\}$*, or*

3. $|N| > 1$ *and for some* $\alpha \in \mathbb{N}$*,* $N = \{2\alpha, 3\alpha, \ldots, (|N|+1)\alpha\}$*,*

Then $\Delta(S)$ *is completely determined. In the first two cases,* $\Delta(S) = N$*; in the last case* $\Delta(S) = N \cup \{\alpha\}$*.*

Beyond Corollary 20, more work is needed to determine $\Delta(S)$. For example, consider the NSCS given by $a_1 = a_2 = 2, b_1 = 7, b_2 = 9$, i.e. $S = \langle 4, 14, 63\rangle$. We have $N = \{5, 7\}$ so applying Theorem 19 gives us $\{1, 5, 7\} \subseteq \Delta(S)$, while a computation with the GAP numericalsgps package (see [9]) shows that $\Delta(S) = \{1, 2, 3, 5, 7\}$.

Lastly, we consider the tame degree $t(S)$. We now prove two lower bounds for $t(S)$. They arise by considering the smallest r_p such that $r_p a_p n_p - n_0 \in S$, and the smallest s_1 such that $s_1 b_1 n_0 - n_p \in S$.

Theorem 21. *Let* $S = \langle n_0, \ldots, n_p\rangle$ *be an NSCS. Set* $r_1 = 1$ *and* $r_i = \lceil \frac{a_{i-1} r_{i-1}}{b_i} \rceil$ *for* $i \in [2, p]$*. Set* $s_p = 1, s_{i-1} = \lceil \frac{b_i s_i}{a_{i-1}} \rceil$ *for* $i \in [2, p]$*. Then*

$$t(S) \geq \max\{(b_1 - a_1)r_1 + (b_2 - a_2)r_2 + \cdots + (b_{p-1} - a_{p-1})r_{p-1} + b_p r_p, b_1 s_1\}.$$

Proof. For the first bound, we set $n = a_p r_p n_p$ and $z = a_p r_p e_p \in \phi^{-1}(n)$. We now set $u = (b_1 r_1, b_2 r_2 - a_1 r_1, b_3 r_3 - a_2 r_2, \ldots, b_p r_p - a_{p-1} r_{p-1}, 0)$. Note the left-first basic chain $u \to \overset{r_1\delta_2}{\cdots} \to (0, b_2 r_2, b_3 r_3 - a_2 r_2, \ldots, b_p r_p - a_{p-1} r_{p-1}, 0) \to \overset{r_2\delta_3}{\cdots} \to (0, 0, b_3 r_3, \ldots, b_p r_p - a_{p-1} r_{p-1}, 0) \to \cdots \to b_p r_p e_{p-1} \to \overset{r_p\delta_p}{\cdots} \to z$. In particular $u \in \phi_0^{-1}(n)$. Note that each $w \in \phi_0^{-1}(n)$ has $|w| > |z|$ and hence $d(w, z) = |w|$. We will now show that $|u| \leq |w|$ for all $w \in \phi_0^{-1}(n)$. First, by Lemma 5, $w_0 \geq b_1 r_1 = u_0$. Now, for all $i \in [1, p-1]$ we must have $u_i < b_{i+1}$ since otherwise $b_{i+1}\lceil \frac{a_i r_i}{b_{i+1}} \rceil - a_i r_i \geq b_{i+1}$, a contradiction. Hence $u - b_1 r_1 e_1$ is a

p-normal factorization. By Corollary 14, $|u - b_1 r_1 e_0| \leq |w - b_1 r_1 e_0|$ and hence $|u| \leq |w|$. Therefore $t_0(n) \geq d(z, \phi_0^{-1}(n)) = d(z, u) = |u| = (b_1 - a_1)r_1 + (b_2 - a_2)r_2 + \cdots + (b_{p-1} - a_{p-1})r_{p-1} + b_p r_p$.

For the second bound, we set $n = b_1 s_1 n_0$ and $z = b_1 s_1 e_0 \in \phi^{-1}(n)$. We now set $u = (0, a_1 s_1 - b_2 s_2, a_2 s_2 - b_3 s_3, \ldots, a_{p-1} s_{p-1} - b_p s_p, a_p s_p)$. Note the right-first basic chain $u \to \overset{s_p \delta'_p}{\cdots} \to (0, a_1 s_1 - b_2 s_2, a_2 s_2 - b_3 s_3, \ldots, a_{p-1} s_{p-1}, 0) \to \cdots \to a_1 s_1 e_1 \to \overset{s_1 \delta'_1}{\cdots} \to z$. In particular $u \in \phi_p^{-1}(n)$. Note that, by Corollary 14, each $w \in \phi^{-1}(n)$ has $|w| \leq |z|$. First, by Lemma 5, $w_p \geq a_p s_p = u_p$. For all $i \in [1, p-1]$ we must have $u_i < a_i$ since otherwise $a_i \lceil \frac{b_{i+1} s_{i+1}}{a_i} \rceil - b_{i+1} s_{i+1} \geq a_i$, a contradiction. Hence $u - a_p s_p e_p$ is a 0-normal factorization. We apply Corollary 13 to conclude that since $u_0 = 0$, also $w'_0 = 0$ for all $w' \in \phi^{-1}(n - a_p s_p n_p)$. Therefore $w_0 = 0$ for all $w \in \phi_i^{-1}(n)$, and hence $d(z, \phi_i^{-1}(n)) = |z| = b_1 s_1$, as desired. □

We have no examples where this inequality is strict. The following examples show that both parts of the bound are necessary. For $S = \langle 165, 176, 208 \rangle$, we compute $t(S) = 27$ while Theorem 21 gives $t(S) \geq \max\{27, 16\}$. For $S = \langle 165, 195, 208 \rangle$, we compute $t(S) = 26$ while Theorem 21 gives $t(S) \geq \max\{18, 26\}$.

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